

THURSTON'S CATACLYSMS FOR ANOSOV REPRESENTATIONS

GUILLAUME DREYER

ABSTRACT. We define deformations for Labourie's Anosov representations that generalize Thurston's cataclysms for hyperbolic structures on surfaces. Given an Anosov representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$, and a maximal geodesic lamination λ in S , we construct shear deformations along the leaves of the geodesic lamination λ endowed with a certain flag decoration that is provided by the associated flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ of the Anosov representation ρ . Such a shear deformation is parametrized by a transverse n -twisted cocycle for the orientation cover $\hat{\lambda}$ of λ . In addition, we establish various geometric properties for these deformations. In particular, we prove a variation formula for the associated length functions ℓ_ρ^i of the Anosov representation ρ .

Let S be a closed, connected, oriented surface of genus $g \geq 2$. In [La], F. Labourie introduced the notion of *Anosov representation* to study elements of the $\mathrm{PSL}_n(\mathbb{R})$ -character variety

$$\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_n(\mathbb{R})) // \mathrm{PSL}_n(\mathbb{R})$$

namely conjugacy classes of homomorphisms $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ from the fundamental group $\pi_1(S)$ to the Lie group $\mathrm{PSL}_n(\mathbb{R})$ (equal to the special linear group $\mathrm{SL}_n(\mathbb{R})$ if n is odd, and to $\mathrm{SL}_n(\mathbb{R})/\{\pm \mathrm{Id}\}$ if n is even). A fundamental property of Anosov representations is the following.

Theorem 1 (Labourie [La]). *Let $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ be an Anosov representation. Then ρ is discrete and injective. In addition, the image $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ of any nontrivial $\gamma \in \pi_1(S)$ is diagonalizable, its eigenvalues are all real with distinct absolute values.*

Important examples of Anosov representations are given by *Hitchin representations*, namely homomorphisms lying in *Hitchin components* $\mathrm{Hit}_n(S)$ of the character variety $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$. A Hitchin component $\mathrm{Hit}_n(S)$ is defined as a component of $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ that contains some *n-Fuchsian representation*, namely some homomorphism $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ of the form

$$\rho = \iota \circ r$$

where: $r : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ is a discrete, injective homomorphism; and $\iota : \mathrm{PSL}_2(\mathbb{R}) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ is the preferred homomorphism defined by the n -dimensional, irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{SL}_n(\mathbb{R})$. These preferred components $\mathrm{Hit}_n(S)$ were identified by N. Hitchin [Hit] who was the first to suggest the interest in studying their elements.

Date: January 30, 2013.

This research was partially supported by the grant DMS-0604866 from the National Science Foundation.

Motivations for studying Hitchin representations find their origin in the case where $n = 2$. Hitchin components $\text{Hit}_2(S)$ then coincide with *Teichmüller components* $\mathcal{T}(S)$ of $\mathcal{R}_{\text{PSL}_2(\mathbb{R})}(S)$, whose elements are of particular interest as they correspond to holonomies of isotopy classes of hyperbolic structures on S . Moreover, every element in $\mathcal{T}(S)$ is a discrete, injective homomorphism, and reversely, any such homomorphism lies in some component $\mathcal{T}(S)$ [We, Mar]. It is a result due to W. Goldman [Gol] that $\mathcal{R}_{\text{PSL}_2(\mathbb{R})}(S)$ possesses exactly two Teichmüller components $\mathcal{T}(S)$, and each of these components $\mathcal{T}(S)$ is known to be homeomorphic to \mathbb{R}^{6g-6} [Th₁, FLP].

In the case where $n \geq 3$, there are one or two Hitchin components $\text{Hit}_n(S)$ in $\mathcal{R}_{\text{PSL}_n(\mathbb{R})}(S)$ depending on whether n is odd or even, and a beautiful result of Hitchin is that each of these components $\text{Hit}_n(S)$ is homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$. Hitchin's proof is based on Higgs bundles theory, and as observed by Hitchin, this complex analysis framework offers no information about the geometry of elements of $\text{Hit}_n(S)$. The first geometric result about Hitchin representations is to due to S. Choi and W. Goldman [ChGo] who showed that, for $n = 3$, the Hitchin component $\text{Hit}_3(S)$ parametrizes the deformation space of *real convex projective structures* on S . A consequence of their work is the faithfulness and the discreteness for the elements of $\text{Hit}_3(S)$.

The powerful Anosov property of Hitchin representations discovered by Labourie [La], and the introduction of *Anosov representations*, have the great advantage to provide a flexible, dynamical-geometric approach to study all Hitchin representations all together, but also other surface group representations. Briefly, given a homomorphism $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, consider the flat twisted bundle $T^1S \times_\rho M = T^1S \times M / \pi_1(S) \rightarrow T^1S$, where T^1S is the unit tangent bundle of S , and where the fibre M is the space of line decomposition of \mathbb{R}^n ; let $(G_t)_{t \in \mathbb{R}}$ on $T^1S \times_\rho M$ be the flow that lifts the geodesic flow $(g_t)_{t \in \mathbb{R}}$ on T^1S . The representation ρ is said to be *Anosov* if there exists a certain section $\sigma_\rho : T^1S \rightarrow T^1S \times_\rho M$ with some Anosov properties for the flow $(G_t)_{t \in \mathbb{R}}$; such a section turns out to be unique: it is the *Anosov section* σ_ρ of the Anosov representation ρ , and it is the central geometric feature of the Anosov representation ρ . In addition, the faithfulness and the discreteness, as well as the fundamental loxodromic property of Theorem 1, all come as consequences of the Anosov dynamics. Because of their properties, Anosov representations constitute a suitable higher-rank version of Teichmüller representations. In particular, we may expect that some concepts and invariants from classical Teichmüller theory extend to the case of Anosov representations.

Results. We extend to Anosov representations *cataclysm deformations* introduced by W. Thurston [Th₂, Bon₁], which themselves generalize (left) *earthquakes* [Th₁, Ker]. Let $r \in \mathcal{T}(S)$ be a Teichmüller representation; and let μ be a *measured lamination* supported in the geodesic lamination $\lambda \subset S$, namely λ is a closed subset foliated by geodesics endowed with a transverse measure supported in λ [Th₁, PeH, Bon₄]. An earthquake is a deformation of the hyperbolic structure on S defined by r via a certain shear operation of the components in the complement $S - \lambda$ along the leaves of the geodesic lamination λ . Such a deformation yields another hyperbolic structure on S of holonomy $\Lambda^\mu r \in \mathcal{T}(S)$. The shear for each component of $S - \lambda$ is determined by the transverse measure μ which parametrizes the earthquake. A feature of earthquakes is that every component of $S - \lambda$ moves in the left direction. Cataclysms are similar to earthquakes, except that the shear

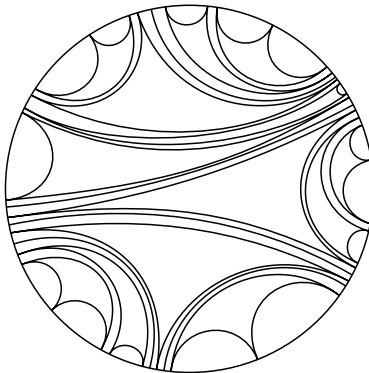


FIGURE 1. The lift of a maximal geodesic lamination λ in S to the universal cover \tilde{S} .

is allowed to simultaneously occur to the left and to the right. In particular, a cataclysm is parametrized by a *transverse cocycle* ε for the geodesic lamination λ [Bon₃, Bon₁], which can be thought as a transverse signed measure that is only finitely additive.

Let $\lambda \subset S$ be a (maximal) geodesic lamination, and let $\hat{\lambda}$ be its orientation cover (in the sense of foliation theory). Cataclysms for Anosov representations are parametrized by the (vector) space of transverse n -twisted cocycles $\mathcal{C}^{\text{Twist}}(\hat{\lambda})$ for the oriented geodesic lamination $\hat{\lambda}$.

Theorem 2. *Let $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ be an Anosov representation. There exist a neighborhood \mathcal{U}^ρ of $0 \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$, and a cataclysm deformation map*

$$\begin{aligned} \Lambda : \mathcal{U}^\rho &\rightarrow \mathcal{R}_{\text{PSL}_n(\mathbb{R})}^{\text{Anosov}}(S) \\ \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) &\mapsto \Lambda^\varepsilon \rho \end{aligned}$$

such that $\Lambda^0 \rho = \rho$. The map Λ is continuous, injective, and coincides with Thurston's cataclysms in the case where $n = 2$.

As for Teichmüller representations, the construction of our cataclysm deformations makes use of the geometry of Anosov representations. Indeed, let $\partial_\infty \tilde{S}$ be the ideal boundary of S ; this object is defined independently of the choice of a hyperbolic metric on S ; see [Ghy, Gro]. The following geometric property will play an central rôle in our construction.

Theorem 3 (Labourie [La]). *Let $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ be an Anosov representation. There exists a unique, Hölder continuous, ρ -equivariant flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$.*

Note that the same invariant flag curve was similarly provided in the case of Hitchin representations by independent work of V. Fock and A. Goncharov [FoGo], who in addition established a certain positivity condition for this flag curve. This approach also proves the faithfulness and the discreteness of Hitchin representations. The point of view of Fock and Goncharov is algebraic geometric and relies on G. Lusztig's notion of positivity [Lu₁, Lu₂]; in particular, it is very different from Labourie's.

The geometric intuition for our cataclysms is to deform an Anosov representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ via a deformation of the associated flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$. Pick a geodesic lamination $\lambda \subset S$. By adding finitely many leaves, we can arrange that λ is *maximal*, namely such that the complement $S - \lambda$ is made of ideal triangles. Let $\tilde{\lambda}$ in the universal cover \tilde{S} of S , that lifts the maximal geodesic lamination $\lambda \subset S$; see Figure 1. The flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ induces an equivariant flag decoration $\mathcal{F}_\rho : \partial_\infty \tilde{\lambda} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ on the set of end-points $\partial_\infty \tilde{\lambda} \subset \partial_\infty \tilde{S}$ of the geodesic lamination $\tilde{\lambda}$; in particular, each ideal triangle in $\tilde{S} - \tilde{\lambda}$ inherits a flag triplet $(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z)) \in \mathrm{Flag}(\mathbb{R}^n)$, on its three vertices $(x, y, z) \in \partial_\infty \tilde{\lambda}$. Similarly as for Teichmüller representations, we define a (equivariant) shear operation for these *flag decorated ideal triangles* along the leaves of the geodesic lamination $\tilde{\lambda}$. The shear for each flag decorated triangle is determined by a transverse n -twisted cocycle $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^{\mathrm{Twist}}(\hat{\lambda})$ for the orientation cover $\hat{\lambda}$. This shear deformation modifies the geometry of the flag curve \mathcal{F}_ρ , and as a result, the Anosov representation ρ .

In [Dr₁], the author generalizes *Thurston's length function* [Th₁, Bon₂, Bon₄] to Anosov representations, which is a fundamental tool in the study of 2 and 3-dimensional hyperbolic manifolds. Among others, one motivation for introducing cataclysms is to analyze the behavior of the lengths ℓ_i^ρ under such deformations. More precisely, fix a maximal geodesic lamination $\lambda \subset S$. Let $\mathcal{C}^H(\hat{\lambda})$ be the (vector) space of transverse cocycles for the orientation cover $\hat{\lambda}$. Given an Anosov representation ρ , the construction in [Dr₁] provides, for every $i = 1, \dots, n$, a linear continuous function $\ell_i^\rho : \mathcal{C}^H(\hat{\lambda}) \rightarrow \mathbb{R}$. We prove the following variational formula.

Theorem 4. *Let $\rho' = \Lambda^\varepsilon \rho$ be a cataclysm deformation of an Anosov representation ρ for some $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^{\mathrm{Twist}}(\hat{\lambda})$ along the maximal geodesic lamination $\lambda \subset S$. Let ℓ_i^ρ and $\ell_i^{\rho'}$ be the associated lengths of ρ and ρ' , respectively. For every transverse cocycle $\alpha \in \mathcal{C}^H(\hat{\lambda})$,*

$$\ell_i^{\rho'}(\alpha) = \ell_i^\rho(\alpha) + \tau(\alpha, \varepsilon_i)$$

where the pairing $\tau : \mathcal{C}^H(\hat{\lambda}) \times \mathcal{C}^H(\hat{\lambda}) \rightarrow \mathbb{R}$ is Thurston's intersection number.

The nature of the above result is essentially algebraic topologic, and a large part of the proof consists of describing certain objects (co)homologically. In particular, a key idea is the homological interpretation of transverse cocycles of $\mathcal{C}^H(\hat{\lambda})$ as elements of the first homology group $H_1(\hat{U})$ where \hat{U} is a preferred open neighborhood for the oriented geodesic lamination $\hat{\lambda}$; see [Bon₁, deRh, RuSu]. In particular, Thurston's intersection number on $\mathcal{C}^H(\hat{\lambda})$, which is a certain type of geometric intersection, turns out to be the same as the classical homology intersection pairing for $H_1(\hat{U})$ (up to a nonzero scalar multiplication); see [Bon₃, Bon₄].

Remarks. A question that this article does not address is the transitivity of cataclysms. In [Dr₂], we define the notion of Anosov representation along a geodesic lamination $\lambda \subset S$, where these considerations find a more natural answer. Cataclysms extend to this class of Anosov representations, and we show the existence of cataclysm paths in this (open) subset. In addition, our analysis gives precise conditions for the existence of such paths in terms of the length functions ℓ_i^ρ introduced in [Dr₁].

Another motivation for studying cataclysms is part of the development of a new system of coordinates for Hitchin components $\text{Hit}_n(S)$. Let us recall Hitchin's result, namely that $\text{Hit}_n(S)$ is diffeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$. Hitchin's parametrization is based on Higgs bundle techniques, and in particular requires the initial choice of a complex structure on S . In a joint work with F. Bonahon [BonDr₁, BonDr₂], we construct a geometric, real analytic parametrization of Hitchin components $\text{Hit}(\mathbb{R}^n)$. One feature of this parametrization is that it is based on topological data only. In essence, our coordinates are an extension of Thurston's shearing coordinates [Th₂, Bon₁] for the Teichmüller space $\mathcal{T}(S)$, combined with Fock-Goncharov's coordinates for the moduli space of positive framed local systems of a punctured surface [FoGo].

1. ANOSOV REPRESENTATIONS

We begin with reviewing some material about Anosov representations. The main objects are the *Anosov section* and the associated *flag curve* of an Anosov representation, which will play a fundamental rôle throughout. Main references for this section are [La, Gui, GuiW₁, GuiW₂].

For convenience, we fix once and for all a hyperbolic metric m_0 on S . The metric m_0 induces a m_0 -geodesic flow $(g_t)_{t \in \mathbb{R}}$ on the unit tangent bundle T^1S ; we refer to the associated orbit space as the m_0 -geodesic foliation \mathcal{F} of T^1S .

1.1. Bundle description(s) of an Anosov representation. Let M be the space of line decompositions of \mathbb{R}^n , namely M is the set of n -tuplets of 1-dimensional subspaces (L_1, \dots, L_n) such that $\mathbb{R}^n = L_1 \oplus \dots \oplus L_n$. Given a homomorphism $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, consider the flat twisted M -bundle

$$T^1S \times_\rho M = T^1\tilde{S} \times M / \pi_1(S) \rightarrow T^1S$$

where: $T^1\tilde{S}$ is the unit tangent bundle of the universal cover of S ; and where the action of $\pi_1(S)$ is defined by the property that

$$\gamma \cdot (\tilde{u}, (L_1, \dots, L_n)) = (\gamma\tilde{u}, (\rho(\gamma)L_1, \dots, \rho(\gamma)L_n))$$

for every $\gamma \in \pi_1(S)$ and $(\tilde{u}, (L_1, \dots, L_n)) \in T^1\tilde{S} \times M$. Via the flat connection, the geodesic flow $(g_t)_{t \in \mathbb{R}}$ on T^1S lifts to a flow $(G_t)_{t \in \mathbb{R}}$ on the total space $T^1S \times_\rho M$. We shall refer to $T^1S \times_\rho M \rightarrow T^1S$ as the *associated M -bundle* of the representation $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$.

A homomorphism $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ is said to be *Anosov* if the associated M -bundle admits a continuous section $\sigma : T^1S \rightarrow T^1S \times_\rho M$, $u \mapsto (V_1(u), \dots, V_n(u))$ satisfying the two following properties:

- (1) The section σ is *flat*, namely if $\tilde{\sigma} : T^1\tilde{S} \rightarrow T^1\tilde{S} \times M$, $\tilde{u} \mapsto (\tilde{V}_1(\tilde{u}), \dots, \tilde{V}_n(\tilde{u}))$ is a lift of σ , then for every $i = 1, \dots, n$, for every $t \in \mathbb{R}$, the two lines $\tilde{V}_i(\tilde{u})$ and $\tilde{V}_i(g_t(\tilde{u})) \subset \mathbb{R}^n$ coincide;
- (2) Let $T^1S \times_\rho \text{End}(\mathbb{R}^n) \rightarrow T^1S$ be the flat twisted $\text{End}(\mathbb{R}^n)$ -bundle, where $\rho(\pi_1(S))$ acts by conjugation on the space of linear endomorphisms $\text{End}(\mathbb{R}^n)$. Let $(\tilde{G}_t)_{t \in \mathbb{R}}$ be the lift to $T^1S \times_\rho \text{End}(\mathbb{R}^n)$ of the geodesic flow $(g_t)_{t \in \mathbb{R}}$. The flat section $\sigma = (V_1, \dots, V_n)$ induces a line splitting $\bigoplus_{1 \leq i, j \leq n} V_i^* \otimes V_j$ of the flat bundle $T^1S \times_\rho \text{End}(\mathbb{R}^n) \rightarrow T^1S$ with the property that each line sub-bundle $V_i^* \otimes V_j \rightarrow T^1S$ is invariant under the action of the flow

$(\bar{G}_t)_{t \in \mathbb{R}}$. We require the restriction of flow $(\bar{G}_t|_{V_i^* \otimes V_j})_{t \in \mathbb{R}}$ to each line bundle $V_i^* \otimes V_j$ to be “Anosov” in the following sense: for every $i \neq j$, there exists a metric $\|\cdot\|$ on $V_i^* \otimes V_j$, and some constants $A \geq 0$ and $a > 0$ such that, $\forall u \in T^1S$, $\forall \psi_u \in V_i^* \otimes V_j(u)$, $\forall t > 0$,

$$\begin{aligned} \text{if } i > j, \quad \|\bar{G}_t \psi_u\|_{g_t(u)} &\leq A e^{-at} \|\psi_u\|_u; \\ \text{if } i < j, \quad \|\bar{G}_{-t} \psi_u\|_{g_{-t}(u)} &\leq A e^{-at} \|\psi_u\|_u. \end{aligned}$$

Remark 5. Here is an alternative description for Anosov representations, which is sometimes easier to work with in practice. Let $\bar{\mathbb{R}}^n = \mathbb{R}^n / \{\pm \text{Id}\}$; note that $\text{PSL}_n(\mathbb{R})$ acts on $\bar{\mathbb{R}}^n$. Given a homomorphism $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, consider the flat twisted $\bar{\mathbb{R}}^n$ -bundle $T^1S \times_\rho \bar{\mathbb{R}}^n \rightarrow T^1S$; and let $(G_t)_{t \in \mathbb{R}}$ be the lift to $T^1S \times_\rho \bar{\mathbb{R}}^n$ of the geodesic flow $(g_t)_{t \in \mathbb{R}}$. Then $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ is an Anosov representation if the bundle $T^1S \times_\rho \bar{\mathbb{R}}^n$ splits as a sum of line sub-bundles $V_1 \oplus \cdots \oplus V_n$ (for the obvious definition of direct sum of lines in this context) with the property that: each line sub-bundle $V_i \rightarrow T^1S$ is invariant under the action of the flow $(G_t)_{t \in \mathbb{R}}$; and the line sub-bundles $V_i \rightarrow T^1S$ satisfy the Anosov property (2). Note that we also abuse the terminology “line bundle” here, as the fibre $V_i(u)$ of $V_i \rightarrow T^1S$ identifies with the quotient of a line of \mathbb{R}^n by $\pm \text{Id}$. However, this discrepancy will have no effect in the following, and we will often think of the components V_i of the Anosov section $\sigma_\rho = (V_1, \dots, V_n)$ as line (sub-)bundles $V_i \rightarrow T^1S$ that are invariant under the action of the flow $(G_t)_{t \in \mathbb{R}}$.

The Anosov property (2) of the flat section $\sigma = (V_1, \dots, V_n)$ has several important consequences, that we now review.

Theorem 6 (Labourie [La]). *Let $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ be an Anosov representation. The associated flat M -bundle $T^1S \times_\rho M \rightarrow T^1S$ admits a unique, flat, continuous section satisfying the Anosov property (2) as above; we shall refer to it as the Anosov section $\sigma_\rho : T^1\tilde{S} \rightarrow T^1\tilde{S} \times M$ of the Anosov representation ρ . In addition, σ_ρ is smooth along the leaves of the geodesic foliation \mathcal{F} of T^1S , and is transversally Hölder continuous.*

The following observation is an easy consequence of the uniqueness of the Anosov section, that we state as a lemma for future reference.

Lemma 7. *Let $\sigma_\rho = (V_1, \dots, V_n)$ be the Anosov section of some Anosov representation ρ , that lifts to $\tilde{\sigma}_\rho = (\tilde{V}_1, \dots, \tilde{V}_n)$. For $\tilde{u} \in T^1\tilde{S}$ projecting to $u \in T^1S$, $\tilde{V}_i(\tilde{u}) = \tilde{V}_{n-i+1}(-\tilde{u})$ as lines of \mathbb{R}^n .*

Proof. Consider the section $\bar{\sigma}_\rho(u) = (V_n(-u), \dots, V_1(-u))$, for $u \in T^1S$. Then $\bar{\sigma}_\rho$ is flat, continuous, and one easily verifies that, for every $t \in \mathbb{R}$, $\bar{\sigma}_\rho(g_t(u)) = (V_n(g_{-t}(-u)), \dots, V_1(g_{-t}(-u)))$. Moreover, since $\sigma_\rho = (V_1, \dots, V_n)$ is the Anosov section, it follows that $\bar{\sigma}_\rho$ also satisfies the Anosov property (2), hence $\bar{\sigma}_\rho = \sigma_\rho$ by uniqueness. \square

A fundamental property of Anosov representations is the following.

Theorem 8 (Labourie [La]). *Let $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ be an Anosov representation. Then ρ is injective and discrete. In addition, the image $\rho(\gamma) \in \text{PSL}_n(\mathbb{R})$ of any nontrivial $\gamma \in \pi_1(S)$ is diagonalizable, and its eigenvalues are all real with distinct absolute values.*

By “ $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ is diagonalizable”, we mean that every lift $\widetilde{\rho(\gamma)} \in \mathrm{SL}_n(\mathbb{R})$ is a diagonalizable matrix. When n is odd, $\mathrm{PSL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})$ and there is no ambiguity. When n is even, $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ admits two lifts $\pm \widetilde{\rho(\gamma)} \in \mathrm{SL}_n(\mathbb{R})$; however, the absolute values of the eigenvalues of $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ are well defined.

We now make the content of Theorem 8 more precise, and also much stronger. Let $\tilde{\sigma}_\rho = (\tilde{V}_1, \dots, \tilde{V}_n)$ that lifts the Anosov section $\sigma_\rho = (V_1, \dots, V_n)$. Pick a nontrivial element $\gamma \in \pi_1(S)$. Let $\widetilde{\rho(\gamma)} \in \mathrm{SL}_n(\mathbb{R})$ that lifts $\rho(\gamma)$. Consider the oriented geodesic $g_\gamma \subset \tilde{S}$ fixed by the isometric action of γ . Let $\tilde{u} \in T^1\tilde{S}$ be a unit vector directing g_γ , and let us set $\tilde{V}_i(g_\gamma) = \tilde{V}_i(\tilde{u}) \subset \mathbb{R}^n$; note that σ_ρ being flat, $\tilde{V}_i(g_\gamma)$ does not depend on the choice of the unit tangent vector \tilde{u} . Since $\gamma\tilde{u} \in g_\gamma$, and $\tilde{V}_i(\gamma\tilde{u}) = \rho(\gamma)\tilde{V}_i(\tilde{u})$ (it is the equivariance property of the lift $\tilde{\sigma}_\rho$), it follows from the above discussion that each line $\tilde{V}_i(g_\gamma)$ is an eigenspace for $\widetilde{\rho(\gamma)}$; let us denote by $\lambda_i^\rho(\gamma) \in \mathbb{R}$ the associated (real) eigenvalue, we shall refer to it as the i -th eigenvalue of $\rho(\gamma)$. A strong consequence of the Anosov property (2) is that, for every nontrivial $\gamma \in \pi_1(S)$,

$$|\lambda_1^\rho(\gamma)| > |\lambda_2^\rho(\gamma)| > \dots > |\lambda_n^\rho(\gamma)|.$$

Finally, let $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}^{\mathrm{Anosov}}(S) \subset \mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ be the set of Anosov representations.

Theorem 9 (Labourie [La]). *The set of Anosov representations $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}^{\mathrm{Anosov}}(S)$ is open in the character variety $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.*

For general results regarding Anosov representations of a surface group to a semisimple Lie group, see [GuiW₁, GuiW₂].

1.2. The flag curve of an Anosov representation. Recall that a (complete) flag F of \mathbb{R}^n consists of a nested sequence of vector subspaces

$$F = F^{(1)} \subset F^{(2)} \dots \subset F^{(n-1)}$$

where each $F^{(i)}$ is a subspace of \mathbb{R}^n of dimension i . Let us denote by $\mathrm{Flag}(\mathbb{R}^n)$ the flag variety of \mathbb{R}^n . A fundamental property of Anosov representations is the existence of an associated equivariant flag curve.

Theorem 10 (Labourie [La]). *Let $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ be an Anosov representation. There exists a unique, continuous, ρ -equivariant flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ which satisfies the following properties:*

- (1) $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ is Hölder continuous;
- (2) \mathcal{F}_ρ is 2-hyperconvex, namely, for every $x \neq y \in \partial_\infty \tilde{S}$,

$$\mathcal{F}_\rho^{(i)}(x) \bigoplus \mathcal{F}_\rho^{(n-i)}(y) = \mathbb{R}^n.$$

By ρ -equivariant, we mean that, for every $\gamma \in \pi_1(S)$, for every $x \in \partial_\infty \tilde{S}$, $\mathcal{F}_\rho(\gamma x) = \rho(\gamma)\mathcal{F}_\rho(x)$.

The existence of the flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ is again a consequence of the Anosov dynamics. \mathcal{F}_ρ derives from the Anosov section $\sigma_\rho : T^1S \rightarrow T^1S \times_\rho M$. In particular, the flag curve \mathcal{F}_ρ and the Anosov section σ_ρ are related as follows. Let $\tilde{\sigma}_\rho = (\tilde{V}_1, \dots, \tilde{V}_n)$ that lifts $\sigma_\rho = (V_1, \dots, V_n)$. For every $\tilde{u} \in T^1\tilde{S}$, let $g \subset \tilde{S}$ be

the oriented geodesic directed by \tilde{u} , and let x_g^+ and $x_g^- \in \partial_\infty \tilde{S}$ be its positive and negative endpoints, respectively. For every $i = 1, \dots, n$,

$$(1) \quad \tilde{V}_i(\tilde{u}) = \mathcal{F}_\rho^{(i)}(x_g^+) \cap \mathcal{F}_\rho^{(n-i+1)}(x_g^-) \subset \mathbb{R}^n$$

with the consequence that

$$\begin{aligned} \mathcal{F}_\rho^{(i)}(x_g^+) &= \tilde{V}_1(\tilde{u}) \subset \tilde{V}_1(\tilde{u}) \oplus \tilde{V}_2(\tilde{u}) \subset \dots \subset \tilde{V}_1(\tilde{u}) \oplus \dots \oplus \tilde{V}_i(\tilde{u}) \\ \mathcal{F}_\rho^{(i)}(x_g^-) &= \tilde{V}_n(\tilde{u}) \subset \tilde{V}_{n-1}(\tilde{u}) \oplus \tilde{V}_n(\tilde{u}) \subset \dots \subset \tilde{V}_{n-i+1}(\tilde{u}) \oplus \dots \oplus \tilde{V}_n(\tilde{u}). \end{aligned}$$

Note that the relation (1) also shows how, by starting from the flag curve \mathcal{F}_ρ , one can reconstruct the Anosov section σ_ρ . Note also that the 2-hyperconvexity of \mathcal{F}_ρ guarantees that $\mathcal{F}_\rho^{(i)}(x_g^+) \cap \mathcal{F}_\rho^{(n-i+1)}(x_g^-) \neq \emptyset$.

Throughout, we will indifferently alternate between the point of view of the Anosov section $\sigma_\rho : T^1 S \rightarrow T^1 S \times_\rho M$, and the one of the flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$, to our liking. The reader should simply keep in mind that manipulating one of the two objects is equivalent to manipulating the other.

We conclude this short review with one last property of the flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$.

Theorem 11 (Labourie [La]). *Let $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ be an Anosov representation along with its associated flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$. The image $\mathcal{F}_\rho(\partial_\infty \tilde{S})$ is the limit set for the action of the subgroup $\rho(\pi_1(S)) \subset \text{PSL}_n(\mathbb{R})$, namely it is the intersection of all $\rho(\pi_1(S))$ -invariant closed subsets in the flag variety $\text{Flag}(\mathbb{R}^n)$.*

2. PRELIMINARIES

2.1. A bunch of estimates. We prove several estimates of which we will make great use throughout.

A *geodesic lamination* $\lambda \subset S$ is a closed subset of S which is a disjoint union of simple (complete) geodesics [PeH, Bon₄]. λ is said to be *maximal* in S if every component of the complement $S - \lambda$ is isometric to an ideal triangle; see Figure 1.

Fix a maximal geodesic lamination $\lambda \subset S$. Let k be a simple arc transverse to λ which does not backtrack so that k intersects each leave of λ at most once. Let d_0 be a component of $k - \lambda$ which does not contain any endpoint of k , and let $g_{d_0}^-$ and $g_{d_0}^+ \subset \lambda$ be the two geodesic leaves passing by the endpoints of d_0 ; note that, λ being maximal, the geodesics $g_{d_0}^-$ and $g_{d_0}^+$ are asymptotic. Consider all components $d \subset k - \lambda$ that are bounded by $g_{d_0}^-$ and $g_{d_0}^+$, namely every component d such that $g_{d_0}^-$ and $g_{d_0}^+$ are both passing by the endpoints of d . As shown on Figure 2, each such subarc d lies in one of two regions delimited by the subarc d_0 and the two leaves $g_{d_0}^-$ and $g_{d_0}^+$. Because of the negative curvature, the two asymptotic geodesics $g_{d_0}^-$ and $g_{d_0}^+$ spread out in the opposite direction. As a result, one of two regions contains finitely many subarcs $d \subset k - \lambda$; we define the *divergence radius* $r(d_0) \in \mathbb{N}$ as the smallest number of subarcs contained in one of the two above regions.

Lemma 12. *For every integer $r \geq 0$, let D_r be the set of components $d \subset k - \lambda$ such that $r(d) = r$. Then*

$$\text{Card}(D_r) \leq 4g - 4$$

where g is the genus of the surface S .

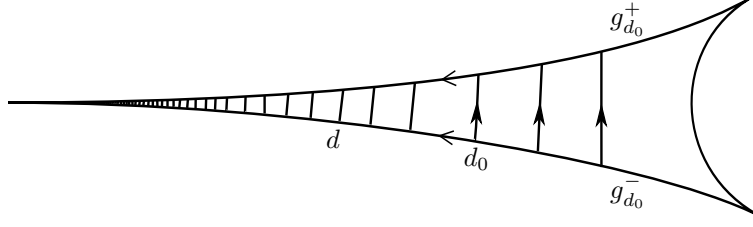


FIGURE 2. A component of $S - \lambda$ that intersects a transverse, oriented arc k .

Lemma 13. *There exist some constant $A > 0$, depending on k , such that, for every component $d \subset k - \lambda$,*

$$\text{length}(d) = O(e^{-Ar(d)}).$$

Proof. Since λ is maximal, and k is simple and does not backtrack, Lemma 12 comes as a consequence of the fact that the complement $S - \lambda$ is made of $4g - 4$ ideal triangles. Lemma 13 follows from classical hyperbolic geometry estimates. See [Bon₁, §1] for details. \square

Let $\sigma_\rho = (V_1, \dots, V_n)$ be the Anosov section of some Anosov representation $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, and let $\tilde{\sigma}_\rho = (\tilde{V}_1, \dots, \tilde{V}_n)$ be a lift. σ_ρ being flat (see §1.1), the lift $\tilde{\sigma}_\rho = (\tilde{V}_1, \dots, \tilde{V}_n)$ associates to every oriented geodesic $g \subset \tilde{S}$ a line decomposition $\tilde{V}_1(g) \oplus \dots \oplus \tilde{V}_n(g)$ of \mathbb{R}^n .

Let $\tilde{\lambda} \subset \tilde{S}$ be the lift of the maximal geodesic lamination $\lambda \subset S$; see Figure 1. Consider a transverse, simple, nonbacktracking, oriented arc k to $\tilde{\lambda}$. Orient positively the leaves of $\tilde{\lambda}$ intersecting k for the transverse orientation determined by the oriented arc k , namely so that the angle between k and every leaf of $\tilde{\lambda}$ is positively oriented. For each component $d \subset k - \tilde{\lambda}$, g_d^+ and $g_d^- \subset \tilde{\lambda}$ denote respectively the oriented geodesics passing by the positive and the negative endpoints of the oriented subarc d . Finally, let dist_{T^1S} be the distance on the unit tangent bundle T^1S ; and let $\text{dist}_{\mathbb{RP}^{n-1}}$ be a metric on \mathbb{RP}^{n-1} .

Lemma 14. *There exist some constant $K > 0$, depending on k and ρ , such that, for every $i = 1, \dots, n$, for every component $d \subset k - \tilde{\lambda}$,*

$$\text{dist}_{\mathbb{RP}^{n-1}}(\tilde{V}_i(g_d^+), \tilde{V}_i(g_d^-)) = O(e^{-Kr(d)}).$$

Proof. Let u_d^- and $u_d^+ \in T^1\tilde{S}$ be the unit vectors based at the positive and the negative endpoints of the oriented subarc $d \subset k - \tilde{\lambda}$ which direct the oriented geodesics g_d^- and g_d^+ , respectively. Note that g_d^- and g_d^+ both converge to, or diverge from their common endpoint. As a result, by compactness of k , for every component $d \subset k - \tilde{\lambda}$, $\text{dist}_{T^1S}(u_d^+, u_d^-) \leq C \text{length}(d)$ for some $C \geq 0$ (depending on k). Since $\tilde{\sigma}_\rho(\tilde{u}) = (\tilde{V}_1(\tilde{u}), \dots, \tilde{V}_n(\tilde{u}))$ depends locally Hölder continuously on $\tilde{u} \in T^1\tilde{S}$, $\text{dist}_{\mathbb{RP}^{n-1}}(\tilde{V}_i(g_d^+), \tilde{V}_i(g_d^-)) \leq C' \text{length}(d)^\mu$ for some $C' \geq 0$ and some $\mu \in (0, 1]$ (both C' and μ depending on k and ρ). An application of Lemma 13 then yields the desired estimate. \square

For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n$, for every oriented geodesic $\underline{g} \subset \tilde{\lambda}$, let us denote by $T_g^\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the linear map which acts on each line $\tilde{V}_i(g) \subset \mathbb{R}^n$ by multiplication by e^{ε_i} .

Let k be a transverse, simple, nonbacktracking, oriented arc to $\tilde{\lambda}$. Orient positively the leaves of $\tilde{\lambda}$ intersecting k for the transverse orientation determined by k . Pick a norm $\|\cdot\|_{\mathbb{R}^n}$ on \mathbb{R}^n ; let $\|\cdot\|$ be the induced norm on the vector space of linear endomorphisms $\text{End}(\mathbb{R}^n)$. Finally, let $\|\cdot\|_{\text{Mat}(\mathbb{R}^n)}$ be a norm on the vector space of square matrices $\text{Mat}_n(\mathbb{R})$.

Lemma 15. *For every component $d \subset k - \tilde{\lambda}$, and for every $\varepsilon \in \mathbb{R}^n$, there exists some constant $K > 0$ depending on k and ρ , such that*

$$\left\| T_{g_d^-}^\varepsilon \circ T_{g_d^+}^{-\varepsilon} - \text{Id} \right\| = O\left(e^{2\|\varepsilon\|_{\mathbb{R}^n} - Kr(d)}\right).$$

Proof. Let \mathcal{B}_d be a basis of unit vectors for $\|\cdot\|_{\mathbb{R}^n}$ which is adapted to the line decomposition $\tilde{V}_1(g_d^-) \oplus \dots \oplus \tilde{V}_n(g_d^-) = \mathbb{R}^n$. Let $\text{Mat}_{\mathcal{B}_d}(T_{g_d^-}^\varepsilon \circ T_{g_d^+}^{-\varepsilon} - \text{Id})$ be the matrix representation of the linear map $T_{g_d^-}^\varepsilon \circ T_{g_d^+}^{-\varepsilon} - \text{Id}$ with respect to the basis \mathcal{B}_d . By an easy calculation, it follows from Lemma 14 that, for every $d \subset k - \tilde{\lambda}$,

$$\left\| \text{Mat}_{\mathcal{B}_d}(T_{g_d^-}^\varepsilon \circ T_{g_d^+}^{-\varepsilon} - \text{Id}) \right\|_{\text{Mat}_n(\mathbb{R})} \leq M e^{2\|\varepsilon\|_{\mathbb{R}^n} - Kr(d)}$$

for some $M \geq 0$ (depending on k and ρ). In addition, k being compact, the set of adapted basis $\{\mathcal{B}_d\}_{d \subset k - \tilde{\lambda}}$ as above lies in some compact subset of $(\mathbb{R}^n)^n$. Hence,

$$\left\| T_{g_d^-}^\varepsilon \circ T_{g_d^+}^{-\varepsilon} - \text{Id} \right\| \leq M' \left\| \text{Mat}_{\mathcal{B}_d}(T_{g_d^-}^\varepsilon \circ T_{g_d^+}^{-\varepsilon} - \text{Id}) \right\|_{\text{Mat}_n(\mathbb{R})}$$

for some $M' \geq 0$ (depending on k and ρ). As a result, for every subarc $d \subset k - \tilde{\lambda}$,

$$\left\| T_{g_d^-}^\varepsilon \circ T_{g_d^+}^{-\varepsilon} - \text{Id} \right\| \leq M'' e^{2\|\varepsilon\|_{\mathbb{R}^n} - Kr(d)}$$

for some $M'' \geq 0$ (depending on k and ρ), which proves the requested estimate. \square

2.2. Transverse cocycles for geodesic laminations. We need to remind the reader of the definition of transverse cocycles for geodesic laminations, along with their main properties. See [Bon₁, Bon₃] for complementary details.

Let $\lambda \subset S$ be a geodesic lamination. A *transverse cocycle* α for λ can be thought as a transverse signed measure for λ that is finitely additive. More precisely, α assigns to every transverse arc k to λ a number $\alpha(k)$, with the property that $\alpha(k) = \alpha(k_1) + \alpha(k_2)$, whenever k_1 and k_2 are two subarcs of k with disjoint interior and such that $k = k_1 \cup k_2$. In addition, α is homotopy invariant, namely $\alpha(k) = \alpha(k')$ if the transverse arc k can be mapped onto the transverse arc k' via a homotopy preserving the leaves of λ .

For the purpose of this paper, we will mostly consider transverse cocycles for the *orientation cover* $\hat{\lambda}$ of some maximal geodesic lamination $\lambda \subset S$. Here, $\hat{\lambda}$ is the orientation cover of λ in the sense of foliation theory, namely $\hat{\lambda}$ is an abstract foliation, which is a 2-cover of the foliation λ , and whose leaves are oriented in a continuous fashion. To be able to talk about transverse cocycles for the orientation cover $\hat{\lambda}$, we need an “ambient surface” for $\hat{\lambda}$, so that we can consider transverse arcs to $\hat{\lambda}$: let $U \subset S$ be an open neighborhood of λ obtained after puncturing the interior of each ideal triangle in $S - \lambda$; the orientation cover $\hat{\lambda} \rightarrow \lambda$ then extends

to a 2-cover $\widehat{U} \rightarrow U$. Therefore, by *transverse cocycle* for $\widehat{\lambda}$, we will always mean a transverse cocycle α for the geodesic lamination $\widehat{\lambda}$ embedded in some open surface \widehat{U} as above. When working with the oriented cover $\widehat{\lambda}$ though, we will often omit to mention the “ambient” surface \widehat{U} and refer to it only when needed. Let $\mathcal{C}^H(\widehat{\lambda})$ be the vector space of transverse cocycles for $\widehat{\lambda}$. It follows from [Bon₃, §5] that the dimension of $\mathcal{C}^H(\widehat{\lambda})$ is finite, with actual dimension equal to $12g - 11$ (since λ is maximal).

Let $k \subset \widehat{U}$ be a transverse, simple, nonbacktracking arc to $\widehat{\lambda}$. Orient k accordingly, namely so that the angle between the oriented arc k and each oriented leaf of $\widehat{\lambda}$ is positively oriented. For every component $d \subset k - \widehat{\lambda}$, k_d is the subarc of k joining the negative endpoint of k to any point contained in d . Pick a norm $\|\cdot\|_{\mathcal{C}^H(\widehat{\lambda})}$ on the vector space $\mathcal{C}^H(\widehat{\lambda})$.

Lemma 16. *There exists some constant $C \geq 0$, depending on k , such that for every $\alpha \in \mathcal{C}^H(\widehat{\lambda})$, and for every component $d \subset k - \widehat{\lambda}$,*

$$\alpha(k_d) \leq C \|\alpha\|_{\mathcal{C}^H(\widehat{\lambda})} (r(d) + 1)$$

where $r(d)$ is the divergence radius of d (see §2.1).

Proof. See [Bon₃, §1]. □

Let $\mathfrak{R} : \widehat{\lambda} \rightarrow \widehat{\lambda}$ be the *orientation reversing involution*. For any $\alpha \in \mathcal{C}^H(\widehat{\lambda})$, $\mathfrak{R}^* \alpha$ is the *pullback* transverse cocycle of α by \mathfrak{R} . We define the vector space of *transverse n -twisted cocycles* for $\widehat{\lambda}$ as

$$\mathcal{C}^{\text{Twist}}(\widehat{\lambda}) = \left\{ \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^H(\widehat{\lambda})^n / \mathfrak{R}^* \varepsilon_i = -\varepsilon_{n-i+1}, \sum \varepsilon_i = 0 \right\}.$$

Lemma 17. *The dimension of the vector space $\mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ is equal to $(n-1)(6g-6) + \lfloor \frac{n-1}{2} \rfloor$.*

Proof. Set $E = \mathcal{C}^H(\widehat{\lambda})$. Let $\tau : E \rightarrow E$, $\alpha \mapsto \mathfrak{R}^* \alpha$ be the pullback endomorphism. τ being an involution, the space E splits as a direct sum $E^+ \oplus E^-$, where E^\pm is the ± 1 -eigenspace. One easily verifies that the subspace E^+ corresponds to the space of transverse cocycles for the maximal geodesic lamination λ , whose dimension is $6g-6$ [Bon₃, §5]; the dimension of E^- is thus $6g-5$. Hence, for every $i = 1, \dots, n$, $\varepsilon_i = \varepsilon_i^+ + \varepsilon_i^-$ for $\varepsilon_i^\pm \in E^\pm$. Since $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$, for every $i = 1, \dots, n$,

$$\mathfrak{R}^* \varepsilon_i = -\varepsilon_{n-i+1}$$

which is equivalent to

$$\begin{cases} \varepsilon_i^+ = -\varepsilon_{n-i+1}^+ \\ \varepsilon_i^- = \varepsilon_{n-i+1}^- \end{cases}$$

By an easy calculation, it follows from the above identities that the dimension of the vector space

$$\left\{ \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^H(\widehat{\lambda})^n / \mathfrak{R}^* \varepsilon_i = -\varepsilon_{n-i+1} \right\}$$

is equal to $(n-1)(6g-6) + \lfloor \frac{n-1}{2} \rfloor + (6g-5)$. Besides, observe that the second condition $\sum \varepsilon_i = 0$ is in fact a condition on the components $\varepsilon_i^- \in E^-$ only; as a result, the dimension of the vector space $\mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ is thus equal to $[(n-1)(6g-6) + \lfloor \frac{n-1}{2} \rfloor + (6g-5)] - (6g-5) = (n-1)(6g-6) + \lfloor \frac{n-1}{2} \rfloor$. □

We conclude these preliminaries with recalling the one-to-one correspondence between transverse cocycles and transverse Hölder distributions for a given geodesic lamination λ [Bon₃, §6]. A *transverse Hölder distribution* for λ assigns to every transverse arc k a Hölder distribution α_k on k , namely α_k is a continuous linear functional defined on the space of Hölder continuous functions; as for transverse cocycles, this assignment is homotopy invariant.

A transverse Hölder distribution α for the geodesic lamination λ defines a transverse cocycle α in a natural fashion: let k be a transverse arc to λ ; consider a Hölder continuous function $\varphi : k \rightarrow \mathbb{R}$ defined on the arc k which is identically equal to 1 on $k \cap \lambda$; the value of the transverse cocycle α at k is set to be $\alpha(k) = \alpha(\varphi)$. Note that this definition is valid for any φ as above as the value $\alpha(\varphi)$ of a transverse Hölder distribution depends only on the values achieved by φ on $k \cap \lambda$ [Bon₃, §4].

Conversely, let k be a transverse arc to λ . Choose an arbitrary orientation for k ; let x_k^+ be the positive endpoint of the oriented arc k ; and let x_d^+ and x_d^- be respectively the positive and negative endpoints of each component $d \subset k - \lambda$. Finally, let k_d be the subarc of d joining the negative endpoint x_k^- of k to an arbitrary point in d . Given a transverse cocycle α for λ , the following formula enables us to reconstruct the corresponding transverse Hölder distribution.

Theorem 18. (*Gap Formula*) *Let α be a transverse Hölder cocycle for a geodesic lamination λ . For every Hölder continuous function $\varphi : k \rightarrow \mathbb{R}$ defined on an oriented arc k transverse to λ , set*

$$\alpha(\varphi) = \alpha(k)\varphi(x_k^+) + \sum_{d \subset k - \lambda} \alpha(k_d)(\varphi(x_d^-) - \varphi(x_d^+))$$

where the indexing d ranges over all components of $k - \lambda$ ($=$ gaps). The above summation is convergent and defines a transverse Hölder distribution for λ .

Proof. See [Bon₃, §5]. □

3. CATACLYSMS

We now tackle the construction of cataclysm deformations for Anosov representations along a maximal geodesic lamination $\lambda \subset S$. The construction will mostly take place in the universal cover $\tilde{S} \supset \tilde{\lambda}$. As in §2.2, $\hat{\lambda}$ will denote the orientation cover of λ .

3.1. Shearing map between two ideal triangles. Consider an Anosov representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ along with its Anosov section $\sigma_\rho = (V_1, \dots, V_n)$. Let P and Q be two ideal triangles in the complement $\tilde{S} - \tilde{\lambda}$. We begin with defining the *shearing map* $\varphi_{PQ} \in \mathrm{SL}_n(\mathbb{R})$ between the triangles P and Q .

Let \mathcal{P}_{PQ} be the set of ideal triangles in $\tilde{S} - \tilde{\lambda}$ lying between P and Q . Let k be a simple, nonbacktracking, oriented arc transverse to $\tilde{\lambda}$ joining a point in the interior of P to a point in the interior of Q . Orient positively the leaves of $\tilde{\lambda}$ intersecting k for the transverse orientation determined by the oriented arc k .

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^{\mathrm{Twist}}(\hat{\lambda})$ be a transverse n -twisted cocycle for the orientation cover $\hat{\lambda}$ of λ ; see §2.2. For every triangle $R \in \mathcal{P}_{PQ}$, let us denote by g_R^- and $g_R^+ \subset \hat{\lambda}$ the two leaves bounding R_j that are the closest to the triangles P and Q , respectively. As in §2.1, the Anosov section $\sigma_\rho = (V_1, \dots, V_n)$ enables us to associate with the oriented geodesics g_R^- and g_R^+ the linear maps $T_{g_R^-}^{\varepsilon(P,R)}$ and

$T_{g_R^+}^{-\varepsilon(P,R)}$, respectively, where $\varepsilon(P, R) \in \mathbb{R}^n$ is defined as follows. As in §2.2, let $U \subset S$ be an open neighborhood of the maximal geodesic lamination λ together with its associated 2-cover $\widehat{U} \supset \widehat{\lambda}$. Consider an oriented arc $k_{PR} \subset \widetilde{U} \subset \widetilde{S}$ transverse to $\widetilde{\lambda}$ joining a point in the interior of the triangle P to a point in the interior of the triangle R . The arc k_{PR} projects to an oriented arc $p(k_{PR}) \subset U \subset S$ which is transverse to λ . Observe that the geodesic lamination $\widehat{\lambda}$ and the surface \widehat{U} being both oriented, $\widehat{\lambda} \subset \widehat{U}$ inherits a well-defined transverse orientation. In particular, the oriented arc $p(k_{PR}) \subset U$ admits a preferred lift $\widehat{p(k_{PR})} \subset \widehat{U}$, namely $\widehat{p(k_{PR})}$ is the unique oriented arc transverse to $\widehat{\lambda}$ that lifts $p(k_{PR})$, and that is oriented accordingly (namely, the angle between the oriented arc $\widehat{p(k_{PR})}$ and each of the oriented leaves of $\widehat{\lambda}$ is positively oriented). We set $\varepsilon(P, R) = \varepsilon(\widehat{p(k_{PR})}) \in \mathbb{R}^n$.

Given a finite subset $\mathcal{P} = \{R_1, R_2, \dots, R_m\} \subset \mathcal{P}_{\mathcal{PQ}}$, where the indexing j of R_j increases as one goes from P to Q , consider the following linear map

$$\varphi_{\mathcal{P}} = T_{g_1^-}^{\varepsilon(P, R_1)} \circ T_{g_1^+}^{-\varepsilon(P, R_1)} \circ T_{g_2^-}^{\varepsilon(P, R_2)} \circ T_{g_2^+}^{-\varepsilon(P, R_2)} \circ \dots \circ T_{g_m^-}^{\varepsilon(P, R_m)} \circ T_{g_m^+}^{-\varepsilon(P, R_m)} \circ T_{g_Q^-}^{\varepsilon(P, Q)}$$

where $g_j^+ = g_{R_j}^+$ and $g_j^- = g_{R_j}^- \subset \widetilde{\lambda}$ are the two leaves bounding R_j that are respectively the closest to the triangles P and Q . Finally, we need a bunch of norms. Pick a norm $\|\cdot\|_{\mathcal{CH}(\widehat{\lambda})}$ on the vector space of transverse cocycles $\mathcal{CH}(\widehat{\lambda})$, and endow the vector space of transverse n -twisted cocycles $\mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ with the norm $\|\varepsilon\| = \max_i \|\varepsilon_i\|_{\mathcal{CH}(\widehat{\lambda})}$ for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$. Likewise, let $\|\cdot\|_{\mathbb{R}^n}$ be the *max norm* on \mathbb{R}^n , namely $\|X\|_{\mathbb{R}^n} = \max_i |x_i|$ for $X = (x_1, \dots, x_n) \in \mathbb{R}^n$, and let $\|\cdot\|$ be the induced norm on $\text{End}(\mathbb{R}^n)$.

Proposition 19. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ small enough,*

$$\lim_{\mathcal{P} \rightarrow \mathcal{P}_{PQ}} \varphi_{\mathcal{P}}$$

exists and is an element of $\text{SL}_n(\mathbb{R})$.

Proof. Set

$$\psi_{\mathcal{P}} = T_{g_1^-}^{\varepsilon(P, R_1)} \circ T_{g_1^+}^{-\varepsilon(P, R_1)} \circ T_{g_2^-}^{\varepsilon(P, R_2)} \circ T_{g_2^+}^{-\varepsilon(P, R_2)} \circ \dots \circ T_{g_m^-}^{\varepsilon(P, R_m)} \circ T_{g_m^+}^{-\varepsilon(P, R_m)}.$$

If the set of ideal triangles \mathcal{P}_{PQ} is finite, there is nothing to prove. We thus assume that \mathcal{P}_{PQ} is an infinite set.

We begin with showing that $\psi_{\mathcal{P}}$ is uniformly bounded.

$$\|\psi_{\mathcal{P}}\| \leq \left\| T_{g_1^-}^{\varepsilon(P, R_1)} \circ T_{g_1^+}^{-\varepsilon(P, R_1)} \right\| \dots \left\| T_{g_m^-}^{\varepsilon(P, R_m)} \circ T_{g_m^+}^{-\varepsilon(P, R_m)} \right\|.$$

By Lemma 15, for every $j = 1, \dots, m$,

$$\left\| T_{g_j^-}^{\varepsilon(P, R_j)} \circ T_{g_j^+}^{-\varepsilon(P, R_j)} - \text{Id} \right\| = O\left(e^{2\|\varepsilon(P, R_j)\|_{\mathbb{R}^n} - Kr(k \cap R_j)}\right)$$

for some $K > 0$ (depending on k and ρ). Therefore, there exists some $M \geq 0$ (depending on k and ρ) such that

$$\|\psi_{\mathcal{P}}\| \leq \prod_{j=1}^m \left(1 + Me^{2\|\varepsilon(P, R_j)\|_{\mathbb{R}^n} - Kr(k \cap R_j)}\right).$$

The convergence of the infinite product on the right-hand side is guaranteed whenever the series $\sum_{j=1}^m e^{2\|\varepsilon(P, R_j)\|_{\mathbb{R}^n} - Kr(k \cap R_j)}$ is convergent. By Lemma 16,

$$\|\varepsilon(P, R_j)\|_{\mathbb{R}^n} = \left\| \varepsilon(\widehat{p(k \cap R_j)}) \right\|_{\mathbb{R}^n} \leq C \|\varepsilon\| \left(r(\widehat{p(k)} \cap \widehat{R_j}) + 1 \right)$$

where $\widehat{R_j} \subset \widehat{U}$ is the lift of the (punctured) triangle $R_j \subset U$. Since $\widehat{U} \rightarrow U$ is a 2-cover, $r(\widehat{p(k)} \cap \widehat{R_j}) \leq r(k \cap R_j)$ where r is the divergence radius (see §2.1). Hence

$$\sum_{j=1}^m e^{2\|\varepsilon(P, R_j)\|_{\mathbb{R}^n} - Kr(k \cap R_j)} \leq \sum_{j=1}^m e^{2C\|\varepsilon\|(r(p(k) \cap R_j) + 1) - Kr(k \cap R_j)}.$$

By Lemma 12, the above series is bounded by finitely many series of the form $\sum_{r=0}^{\infty} e^{2C\|\varepsilon\|(r+1) - Kr}$; this implies that $\|\psi_{\mathcal{P}}\|$ is uniformly bounded whenever $\|\varepsilon\| < K/2C$.

We now prove that $\psi_{\mathcal{P}}$ converges as \mathcal{P} goes to \mathcal{P}_{PQ} . Let \mathcal{P}_m be an increasing sequence of finite ideal triangles converging to \mathcal{P}_{PQ} with $\text{Card}(\mathcal{P}_m) = m$. Consider the maps $\psi_{\mathcal{P}_m}$ and $\psi_{\mathcal{P}_{m+1}}$. Since \mathcal{P}_{m+1} contains one more triangle R than \mathcal{P}_m , and $\|\psi_{\mathcal{P}}\|$ is uniformly bounded,

$$\psi_{\mathcal{P}_m} = \psi_{\mathcal{P}} \psi_{\mathcal{P}'} \text{ and } \psi_{\mathcal{P}_{m+1}} = \psi_{\mathcal{P}} \circ T_{g_R^-}^{\varepsilon(P, R)} \circ T_{g_R^+}^{-\varepsilon(P, R)} \circ \psi_{\mathcal{P}'}$$

where $\mathcal{P}_m = \mathcal{P} \cup \mathcal{P}'$. Hence, by Lemma 15,

$$\begin{aligned} \|\psi_{\mathcal{P}_{m+1}} - \psi_{\mathcal{P}_m}\| &\leq \|\psi_{\mathcal{P}}\| \left\| T_{g_R^-}^{\varepsilon(P, R)} \circ T_{g_R^+}^{-\varepsilon(P, R)} - \text{Id} \right\| \|\psi_{\mathcal{P}'}\| \\ &\leq M' e^{2\|\varepsilon(P, R)\|_{\mathbb{R}^n} - Kr(k \cap R)} \end{aligned}$$

for some $M' \geq 0$ (depending on k and ρ). Since \mathcal{P}_{PQ} is an infinite set, Lemma 12 implies that $\lim_{m \rightarrow \infty, R \in \mathcal{P}_m} r(k \cap R) = \infty$. In particular, the sequence $\psi_{\mathcal{P}_m}$ is Cauchy, and thus convergent whenever $\|\varepsilon\| < K/2C$. In fine, $\lim_{\mathcal{P} \rightarrow \mathcal{P}_{PQ}} \psi_{\mathcal{P}}$, and so $\lim_{\mathcal{P} \rightarrow \mathcal{P}_{PQ}} \varphi_{\mathcal{P}} = \psi_{\mathcal{P}} \circ T_{g_Q^-}^{\varepsilon(P, Q)}$, are well-defined maps for $\varepsilon \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ small enough. \square

The above proof also gives the following estimate, that will come handy later.

Corollary 20. *There exists some constant $B > 0$, depending on k and ρ , such that, for $\varepsilon \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ small enough,*

$$\varphi_{PQ} = \psi_{PQ} \circ T_{g_Q^-}^{\varepsilon(P, Q)}$$

where $\psi_{PQ} = \text{Id} + \mathcal{O}\left(\sum_{R \in \mathcal{P}_{PQ}} e^{-Br(k \cap R)}\right)$.

We emphasize the fact that the shearing map $\varphi_{PQ} = \varphi_{PQ}^{\varepsilon} \in \text{SL}_n(\mathbb{R})$ is determined by the transverse n -twisted cocycle $\varepsilon \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$.

3.2. Composition of shearing maps. Let $\varphi_{\mathcal{P}}$ be as in Proposition 19. Shearing maps φ_{PQ} satisfy the following properties.

Theorem 21. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ small enough, for every plaques P, Q, R of $\widetilde{S} - \widetilde{\lambda}$, the map $\varphi_{\mathcal{P}}$ converges to a linear map $\varphi_{PQ} \in \text{SL}_n(\mathbb{R})$, as \mathcal{P} tends to \mathcal{P}_{PQ} . In addition, $\varphi_{QP} = \varphi_{PQ}^{-1}$ and $\varphi_{PR} = \varphi_{PQ} \varphi_{QR}$.*

Proof of Theorem 21. The demonstration will require several steps. In particular, we will consider an alternative description for the shearing map φ_{PQ} for which the composition property will be straightforward.

Let k be a transverse, simple, nonbacktracking, oriented arc to $\tilde{\lambda}$ joining a point in the interior of P to a point in the interior of Q . For every integer $r > 0$, let \mathcal{P}_{PQ}^r be the finite set of triangles $R \in \mathcal{P}_{PQ}$ such that the divergence radius $r(k \cap R) \leq r$; see §2.1. Index the elements of \mathcal{P}_{PQ}^r as R_1, R_2, \dots, R_m so that the indexing j of R_j increases as one goes from P to Q . For every $j = 1, \dots, m$, pick a geodesic h_j separating the interior of R_j from the interior of R_{j+1} . Pick also a geodesic h_0 between P and R_1 , and a geodesic h_m between R_m and Q , and orient positively the h_j for the transverse orientation determined by the oriented arc k .

Set

$$\varphi_{PQ}^r = T_{h_0}^{\varepsilon(P, R_1)} \circ T_{h_1}^{\varepsilon(R_1, R_2)} \circ T_{h_2}^{\varepsilon(R_2, R_3)} \circ \dots \circ T_{h_m}^{\varepsilon(R_m, Q)}.$$

Proposition 22. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough, φ_{PQ}^r is convergent as r tends to ∞ and*

$$\lim_{r \rightarrow \infty} \varphi_{PQ}^r = \varphi_{PQ}.$$

Proof. We will first estimate the difference between the map $\psi_{\mathcal{P}_{PQ}^r}$ of the proof of Proposition 19 and the map $\psi_{PQ}^r = \varphi_{PQ}^r \circ T_{h_m}^{-\varepsilon(PQ)}$. By reordering the terms in the expression of ψ_{PQ}^r , we have

$$\psi_{PQ}^r = T_{h_0}^{\varepsilon(P, R_1)} \circ T_{h_1}^{-\varepsilon(P, R_1)} \circ T_{h_1}^{\varepsilon(P, R_2)} \circ T_{h_2}^{-\varepsilon(P, R_2)} \circ \dots \circ T_{h_{m-1}}^{\varepsilon(P, R_m)} \circ T_{h_m}^{-\varepsilon(P, R_m)}$$

and as previously,

$$\psi_{\mathcal{P}_{PQ}^r} = T_{g_1^-}^{\varepsilon(P, R_1)} \circ T_{g_1^+}^{-\varepsilon(P, R_1)} \circ T_{g_2^-}^{\varepsilon(P, R_2)} \circ T_{g_2^+}^{-\varepsilon(P, R_2)} \circ \dots \circ T_{g_m^-}^{\varepsilon(P, R_m)} \circ T_{g_m^+}^{-\varepsilon(P, R_m)}.$$

Note that the map ψ_{PQ}^r is obtained from $\psi_{\mathcal{P}_{PQ}^r}$ by replacing each term $T_{g_j^-}^{\varepsilon(P, R_j)} \circ T_{g_j^+}^{-\varepsilon(P, R_j)}$ by $T_{h_{j-1}}^{\varepsilon(P, R_j)} \circ T_{h_j}^{-\varepsilon(P, R_j)}$.

Consider the *train track* [PeH, Bon₄] associated to the transverse arc k that contains the maximal geodesic lamination $\tilde{\lambda}$.

Lemma 23. *The two geodesics g_j^+ and $g_{j+1}^- \subset \tilde{\lambda}$ follow the same edge-path of length $2r$ in the train track associated to the transverse arc k .*

Proof. If they do not, there exists an ideal triangle R between R_j and R_{j+1} whose sides g_R^- and g_R^+ follow the same edge-paths of length $2r$ as g_j^+ and g_{j+1}^- , respectively. g_R^- and g_R^+ being asymptotic, it implies that g_R^- and g_R^+ must follow the same edge-path of length r , hence $R \in \mathcal{P}_{PQ}^r$ which contradicts the assumption. \square

In one hand, since the geodesic h_j lies between g_j^+ and g_{j+1}^- , it follows the same edge-path of length $2r$ in the train track. In particular, the distance between any two of these three geodesics is thus a $O(e^{-Ar})$ for some $A \geq 0$ (depending on k). On the other hand, by Lemma 13, the distance between g_j^- and g_j^+ is a $O(e^{-Ar(k \cap R_j)})$. Recall that $R_j \in \mathcal{P}_{PQ}^r$, so $r(k \cap R_j) \leq r$. The above discussion implies that the distance between h_j and h_{j+1} is also a $O(e^{-Ar(k \cap R_j)})$. Following the arguments

in the proof of Lemma 15, the previous estimates show that we can find some constants $M \geq 0$ and $K > 0$ (both depending on k and ρ) for which, for every j ,

$$\left\| T_{h_j}^{\varepsilon(P, R_j)} \circ T_{h_{j+1}}^{-\varepsilon(P, R_j)} - \text{Id} \right\| \leq M e^{2C\|\varepsilon\|(r(k \cap R_j)+1)} e^{-Kr(k \cap R_j)}$$

and

$$\left\| T_{g_j^-}^{\varepsilon(P, R_j)} \circ T_{g_j^+}^{-\varepsilon(P, R_j)} - \text{Id} \right\| \leq M e^{2C\|\varepsilon\|(r(k \cap R_j)+1)} e^{-Kr(k \cap R_j)}.$$

Let ψ be any map obtained from $\psi_{\mathcal{P}_{PQ}^r}$ by replacing some of the m terms $T_{g_j^-}^{\varepsilon(P, R_j)} \circ T_{g_j^+}^{-\varepsilon(P, R_j)}$ by $T_{h_{j-1}}^{\varepsilon(P, R_j)} \circ T_{h_j}^{-\varepsilon(P, R_j)}$ or by the identity. As in the proof of Proposition 19, it follows from the latter estimates that

$$\begin{aligned} \log \|\psi\| &= O \left(\sum_{j=1}^m e^{2C\|\varepsilon\|(r(k \cap R_j)+1)} e^{-Kr(k \cap R_j)} \right) \\ &= O \left(\sum_{r=0}^{\infty} e^{2C\|\varepsilon\|(r+1)} e^{-Kr} \right). \end{aligned}$$

Consequently, the norm of such a map ψ is uniformly bounded, whenever $\|\varepsilon\| \leq K/2C$.

Let ψ_l be obtained from $\psi_{\mathcal{P}_{PQ}^r}$ by replacing each $T_{g_j^-}^{\varepsilon(P, R_j)} \circ T_{g_j^+}^{-\varepsilon(P, R_j)}$ with $j \leq l$ by $T_{h_{j-1}}^{\varepsilon(P, R_j)} \circ T_{h_j}^{-\varepsilon(P, R_j)}$, so that $\psi_0 = \psi_{\mathcal{P}_{PQ}^r}$ and $\psi_m = \psi_{PQ}^r$. Again, as in the proof of Proposition 19, we estimate the difference between ψ_{l-1} and ψ_l . Let us rewrite $\psi_{l-1} = \psi \circ T_{g_l^-}^{\varepsilon(P, R_l)} \circ T_{g_l^+}^{-\varepsilon(P, R_l)} \circ \psi'$ and $\psi_l = \psi \circ T_{h_{l-1}}^{\varepsilon(P, R_l)} \circ T_{h_l}^{-\varepsilon(P, R_l)} \circ \psi'$, where ψ and ψ' are obtained from replacing some $T_{g_j^-}^{\varepsilon(P, R_j)} \circ T_{g_j^+}^{-\varepsilon(P, R_j)}$ by $T_{h_{j-1}}^{\varepsilon(P, R_j)} \circ T_{h_j}^{-\varepsilon(P, R_j)}$ or the identity. As observed above, $\|\psi\|$ and $\|\psi'\|$ are uniformly bounded. Hence

$$\begin{aligned} \|\psi_{l-1} - \psi_l\| &\leq \|\psi\| \left\| T_{g_l^-}^{\varepsilon(P, R_l)} \circ T_{g_l^+}^{-\varepsilon(P, R_l)} - T_{h_{l-1}}^{\varepsilon(P, R_l)} \circ T_{h_l}^{-\varepsilon(P, R_l)} \right\| \|\psi'\| \\ &= O \left(e^{4C\|\varepsilon\|(r+1)-Kr} \right). \end{aligned}$$

Therefore,

$$\left\| \psi_{PQ}^r - \psi_{\mathcal{P}_{PQ}^r} \right\| = \|\psi_m - \psi_0\| \leq m O \left(e^{4C\|\varepsilon\|(r+1)-Kr} \right) = O \left(r e^{4C\|\varepsilon\|(r+1)-Kr} \right)$$

since $m = \text{Card}(\mathcal{P}_{PQ}^r) = O(r)$ by Lemma 12. We conclude that ψ_{PQ}^r and $\psi_{\mathcal{P}_{PQ}^r}$ have the same limit as r tends to ∞ whenever $\|\varepsilon\| < K/4C$ (recall that $\psi_{\mathcal{P}_{PQ}^r}$ converges whenever $\|\varepsilon\| < K/2C$). At last, observe that h_m converges to g_Q^- , which implies that both $\varphi_{PQ}^r = \psi_{PQ}^r \circ T_{h_m}^{\varepsilon(P, Q)}$ and $\varphi_{\mathcal{P}_{PQ}^r} = \psi_{\mathcal{P}_{PQ}^r} \circ T_{g_Q^-}^{\varepsilon(P, Q)}$ both converge to the same limit φ_{PQ} . \square

Corollary 24. *Let k be a transverse, simple, nonbacktracking, oriented arc to $\tilde{\lambda}$. For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\tilde{\lambda})$ small enough, for every triangles $P, Q, R \subset \tilde{S} - \tilde{\lambda}$ intersecting the arc k , $\varphi_{QP} = (\varphi_{PQ})^{-1}$ and $\varphi_{PR} = \varphi_{PQ}\varphi_{QR}$.*

Proof. Let k be as above. First, suppose that the oriented arc k intersects the triangles P, Q and R in this order. Then, it is a straightforward consequence of Proposition 22 that $\varphi_{PR} = \varphi_{PQ}\varphi_{QR}$.

Likewise, let $\mathfrak{R}(k)$ be the arc k , but oriented in the opposite direction; in particular, the oriented arc $\mathfrak{R}(k)$ first intersects Q , then P . Orient positively the leaves of $\tilde{\lambda}$ that intersects $\mathfrak{R}(k)$ for the transverse orientation determined by the oriented arc $\mathfrak{R}(k)$. Then, with (exactly) the same notations as in Proposition 22, we have that $\varphi_{QP} = \lim_{r \rightarrow \infty} \varphi_{QP}^r$, where

$$\varphi_{QP}^r = T_{\mathfrak{R}(h_m)}^{\varepsilon(Q, R_m)} \circ T_{\mathfrak{R}(h_{m-1})}^{\varepsilon(R_m, R_{m-1})} \circ \dots \circ T_{\mathfrak{R}(h_1)}^{\varepsilon(R_2, R_1)} \circ T_{\mathfrak{R}(h_m)}^{\varepsilon(R_1, P)}$$

with the difference that each oriented geodesic h_j has been replaced by $\mathfrak{R}(h_j)$, which denotes the same geodesic h_j but oriented in the opposite direction. Let us consider the general term $T_{\mathfrak{R}(h_j)}^{\varepsilon(R_{j+1}, R_j)}$. Since $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$, it follows from the definition of $\varepsilon(R_{j+1}, R_j) = (\varepsilon_1(R_{j+1}, R_j), \dots, \varepsilon_n(R_{j+1}, R_j))$ (see §3.1) that

$$\begin{aligned} \varepsilon(R_{j+1}, R_j) &= (\mathfrak{R}^* \varepsilon)(R_j, R_{j+1}) \\ &= (-\varepsilon_n(R_j, R_{j+1}), \dots, -\varepsilon_1(R_j, R_{j+1})). \end{aligned}$$

Moreover, by Lemma 7, the line decomposition associated with the oriented geodesic $\mathfrak{R}(h_j)$ is $\tilde{V}_n(h_j) \oplus \dots \oplus \tilde{V}_1(h_j) = \mathbb{R}^n$. As a result, by definition of the linear map $T_{\mathfrak{R}(h_j)}^{\varepsilon(R_{j+1}, R_j)}$ (see §2.1),

$$T_{\mathfrak{R}(h_j)}^{\varepsilon(R_{j+1}, R_j)} = \left(T_{h_j}^{\varepsilon(R_j, R_{j+1})} \right)^{-1}$$

and we conclude immediately that $\varphi_{QP} = \lim_{r \rightarrow \infty} \varphi_{QP}^r = (\varphi_{PQ})^{-1}$. The asserted result follows from these two special cases. \square

In all previous statements, the size of the transverse n -twisted cocycle $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ depends on the considered transverse arc k and on the Anosov representation ρ .

Lemma 25. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough, for every triangles P, Q, R of $\tilde{S} - \tilde{\lambda}$, the map $\varphi_{\mathcal{P}}$ converges to a linear map $\varphi_{PQ} \in \text{SL}_n(\mathbb{R})$, as \mathcal{P} tends to \mathcal{P}_{PQ} . In addition, $\varphi_{QP} = \varphi_{PQ}^{-1}$ and $\varphi_{PR} = \varphi_{PQ} \varphi_{QR}$.*

Proof. Pick in the surface S finitely many transverse arcs k_1, \dots, k_N to λ , and such that each component of $S - \lambda$ meets at least one of the k_i . Given two triangles P and Q in $\tilde{S} - \tilde{\lambda}$, there is a finite sequence of triangles $R_0 = P, R_1, \dots, R_N, R_{N+1} = Q$ such that each R_j separates R_{j-1} from R_{j+1} , and such that R_j and R_{j+1} meets the same lift \tilde{k}_{i_j} . Choose ε small enough so that the convergence of the $\varphi_{R_j, R_{j+1}}$ is guaranteed for every $j = 1, \dots, N$. It follows that $\varphi_{PQ} = \lim_{\mathcal{P} \rightarrow \mathcal{P}_{PQ}} \varphi_{\mathcal{P}}$ exists, and is equal to $\varphi_{R_0 R_1} \varphi_{R_1 R_2} \dots \varphi_{R_N R_{N+1}}$. \square

This achieves the proof of Theorem 21. \square

As a consequence of Lemma 25, the condition “ $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough” becomes a condition depending on the Anosov representation ρ only.

3.3. Cataclysm deformations. Let $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ be a transverse n -twisted cocycle sufficiently small. Fix a triangle $P_0 \subset \tilde{S} - \tilde{\lambda}$. The ε -cataclysm deformation of the Anosov representation ρ along the maximal geodesic lamination λ is the homomorphism $\Lambda^\varepsilon \rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ defined as follows: for every $\gamma \in \pi_1(S)$,

$$\Lambda^\varepsilon \rho(\gamma) = \varphi_{P_0 \gamma P_0} \circ \rho(\gamma)$$

where $\varphi_{P_0\gamma P_0}$ is the shearing map between the two triangles P_0 and $\gamma P_0 \subset \tilde{S} - \tilde{\lambda}$; see §3.1.

We must verify that $\Lambda^\varepsilon \rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ is a group homomorphism. Put $\rho' = \Lambda^\varepsilon \rho$. By definition of the shearing map φ_{PQ} , one easily verifies that it satisfies the following equivariant property: for every $\gamma \in \pi_1(S)$, for every $P, Q \subset \tilde{S} - \tilde{\lambda}$, $\varphi_{\gamma P \gamma Q} = \rho(\gamma) \circ \varphi_{PQ} \circ \rho(\gamma)^{-1}$. Thus, for every $\gamma_1, \gamma_2 \in \pi_1(S)$,

$$\begin{aligned} \rho'(\gamma_1 \gamma_2) &= \varphi_{P_0 \gamma_1 \gamma_2 P_0} \circ \rho(\gamma_1 \gamma_2) \\ &= \varphi_{P_0 \gamma_1 P_0} \circ \varphi_{\gamma_1 P_0 \gamma_1 \gamma_2 P_0} \circ \rho(\gamma_1) \circ \rho(\gamma_2) \\ &= \varphi_{P_0 \gamma_1 P_0} \circ \rho(\gamma_1) \circ \varphi_{P_0 \gamma_2 P_0} \circ \rho(\gamma_2) \\ &= \rho'(\gamma_1) \rho'(\gamma_2). \end{aligned}$$

Note that a different choice of triangle $P_0 \subset \tilde{S} - \tilde{\lambda}$ yields another homomorphism ρ'' that is conjugate to the previous ρ' ; in particular, $\rho' = \Lambda^\varepsilon \rho$ defines without any ambiguity a point in the character variety $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$.

Recall that the set of Anosov representations $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}^{\mathrm{Anosov}}(S)$ is open in the character variety $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ [La, GuiW₂]. We can now state the main result of this section.

Theorem 26. *Let $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ be an Anosov representation. There exist an open neighborhood \mathcal{U}^ρ of $0 \in \mathcal{C}^{\mathrm{Twist}}(\hat{\lambda})$, and a continuous, injective cataclysm deformation map*

$$\begin{aligned} \Lambda : \mathcal{U}^\rho &\rightarrow \mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}^{\mathrm{Anosov}}(S) \\ \varepsilon &\mapsto \Lambda^\varepsilon \rho \end{aligned}$$

such that $\Lambda^0 \rho = \rho$.

We shall refer to the transverse n -twisted cocycle $\varepsilon \in \mathcal{C}^{\mathrm{Twist}}(\hat{\lambda})$ as the *shear parameter* of the cataclysm deformation $\Lambda^\varepsilon \rho$; it determines the “magnitude” of the cataclysm.

The injectivity of the cataclysm map $\Lambda : \mathcal{U}^\rho \rightarrow \mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}^{\mathrm{Anosov}}(S)$ will be proved in §5.1; see Corollary 36.

4. CATACLYSMS AND FLAG CURVES

We now study the effect of a cataclysm deformation on the associated equivariant flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ of some Anosov representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$.

Let $\rho' = \Lambda^\varepsilon \rho$ be a ε -cataclysm deformation of ρ along a maximal geodesic lamination $\lambda \subset S$ for some transverse n -twisted cocycle $\varepsilon \in \mathcal{C}^{\mathrm{Twist}}(\hat{\lambda})$ small enough. Fix an ideal triangle $P_0 \subset \tilde{S} - \tilde{\lambda}$, and consider the equivariant family of shearing maps $\{\varphi_{P_0 P}\}_{P \subset \tilde{S} - \tilde{\lambda}} \subset \mathrm{SL}_n(\mathbb{R})$; see §3.1. Let $\mathcal{V}_\lambda \subset \partial_\infty \tilde{S}$ be the set of vertices of the ideal triangles in $\tilde{S} - \tilde{\lambda}$; note that the set $\mathcal{V}_\lambda \subset \partial_\infty \tilde{S}$ is $\pi_1(S)$ -invariant. For every $x \in \mathcal{V}_\lambda$, let $P \subset \tilde{S} - \tilde{\lambda}$ be an ideal triangle whose x is a vertex, and let $\varphi_{P_0 P} \in \mathrm{PSL}_n(\mathbb{R})$ be the associated shearing map. Set

$$\mathcal{F}'_\rho(x) = \varphi_{P_0 P}(\mathcal{F}_\rho(x))$$

where $\mathcal{F}_\rho(x)$ is the image of the vertex $x \in \mathcal{V}_\lambda$ by the flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$.

Lemma 27. *The above relation defines a ρ' -equivariant flag map $\mathcal{F}'_\rho : \mathcal{V}_\lambda \rightarrow \text{Flag}(\mathbb{R}^n)$.*

Proof. There is an ambiguity in the definition of $\mathcal{F}'_\rho(x)$ as a vertex $x \in \mathcal{V}_\lambda$ may belong to several ideal triangles in $\tilde{S} - \tilde{\lambda}$. Suppose that there is another triangle $Q \subset \tilde{S} - \tilde{\lambda}$ whose x is a vertex, and let us compare the images $\varphi_{P_0P}(\mathcal{F}_\rho(x))$ and $\varphi_{P_0Q}(\mathcal{F}_\rho(x))$. By applying Lemma 25,

$$\varphi_{P_0Q}(\mathcal{F}_\rho(x)) = \varphi_{P_0P}(\varphi_{PQ}(\mathcal{F}_\rho(x))).$$

Observe that, since P and Q share the same vertex $x \in \mathcal{V}_\lambda$, any ideal triangle R between P and Q admits x as one of its vertices. Therefore, for every such triangle R , both the linear maps $T_{g_R^-}^{\varepsilon(P,R)}$ and $T_{g_R^+}^{-\varepsilon(P,R)}$ (see §3.1) fix the flag $\mathcal{F}_\rho(x)$. It follows from the definition of the shearing map φ_{PQ} that $\varphi_{PQ}(\mathcal{F}_\rho(x)) = \mathcal{F}_\rho(x)$, and thus that $\varphi_{P_0Q}(\mathcal{F}_\rho(x)) = \varphi_{P_0P}(\mathcal{F}_\rho(x))$. The ρ' -equivariance comes as a straightforward consequence of the equivariance properties of the flag curve \mathcal{F}_ρ , and of the family $\{\varphi_{P_0P}\}_{P \subset \tilde{S} - \tilde{\lambda}} \subset \text{SL}_n(\mathbb{R})$. \square

Let $\partial_\infty \tilde{\lambda} \subset \partial_\infty \tilde{S}$ be the set of ideal endpoints of all leaves contained in the geodesic lamination $\tilde{\lambda} \subset \tilde{S}$; note that $\partial_\infty \tilde{\lambda} \supset \mathcal{V}_\lambda$. We wish to extend the previous flag map $\mathcal{F}'_\rho : \mathcal{V}_\lambda \rightarrow \text{Flag}(\mathbb{R}^n)$ to a flag map $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$. To this end, we generalize the way to define $\mathcal{F}'_\rho : \mathcal{V}_\lambda \rightarrow \text{Flag}(\mathbb{R}^n)$ of Lemma 27.

Let $g \subset \tilde{\lambda}$ be a geodesic leaf. Consider a triangle $P \subset \tilde{S} - \tilde{\lambda}$ such that there exists a simple, nonbacktracking, oriented arc k transverse to $\tilde{\lambda}$ joining a point in the interior of P_0 to a point in the interior of P , and intersecting the leaf g . Orient positively the leaves of $\tilde{\lambda}$ for the transverse orientation defined by the oriented arc k . Let \mathcal{P}_{P_0g} be the set of ideal triangles of $\tilde{S} - \tilde{\lambda}$ lying between the ideal triangle P_0 and the geodesic g . Similarly as in §3.1, set

$$\psi_{\mathcal{P}} = \prod_{j=1}^m \left(T_{g_j^-}^{\varepsilon(P_0, R_j)} \circ T_{g_j^+}^{-\varepsilon(P_0, R_j)} \right)$$

where $\mathcal{P} = \{R_1, R_2, \dots, R_m\} \subset \mathcal{P}_{P_0g}$ is a finite subset, and where the indexing j of R_j increases as one goes from P_0 to g .

Lemma 28. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough, for every leaf $g \subset \tilde{\lambda}$,*

$$\psi_{P_0g} = \lim_{\mathcal{P} \rightarrow \mathcal{P}_{P_0g}} \psi_{\mathcal{P}}$$

exists and is an element of $\text{SL}_n(\mathbb{R})$.

Proof. By Theorem 21, whenever $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ is small enough, for every $P \subset \tilde{S} - \tilde{\lambda}$, the linear map $\psi_{P_0P} \in \text{SL}_n(\mathbb{R})$ is well defined. Moreover, similarly as in Corollary 20, the following estimate

$$\psi_{P_0P} = O \left(\sum_{R \in \mathcal{P}_{P_0P}} e^{-Br(k \cap R)} \right)$$

holds for some $B > 0$ (depending on k and on ρ). As a result,

$$\psi_{P_0g} = O \left(\sum_{R \in \mathcal{P}_{P_0g}} e^{-Br(k \cap R)} \right)$$

which implies that ψ_{P_0g} is convergent. \square

Having defined the family of linear maps $\{\psi_{P_0g}\}_{g \subset \tilde{\lambda}}$, for every $x \in \partial_\infty \tilde{\lambda}$, set

$$\mathcal{F}'_\rho(x) = \psi_{P_0g}(\mathcal{F}_\rho(x))$$

where $g \subset \tilde{\lambda}$ is a geodesic whose x is an endpoint.

Lemma 29. *The above relation defines a ρ' -equivariant flag map $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$ that extends the flag map $\mathcal{F}'_\rho : \mathcal{V}_\lambda \rightarrow \text{Flag}(\mathbb{R}^n)$ of Lemma 27.*

Proof. Again, we must check that there is no ambiguity in the definition of the map $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$, and that the newly defined flag map coincides with the map $\mathcal{F}'_\rho : \mathcal{V}_\lambda \rightarrow \text{Flag}(\mathbb{R}^n)$ of Lemma 27.

Observe that if $x \in \partial_\infty \tilde{\lambda} - \mathcal{V}_\lambda$, there is a unique geodesic $g \subset \tilde{\lambda}$ with x as an endpoint. The above relation thus associates to such a point x a unique flag $\mathcal{F}'_\rho(x) \in \text{Flag}(\mathbb{R}^n)$.

Now, suppose that $x \in \mathcal{V}_\lambda$, namely g is one of two edges g_P^- and g_P^+ bounding some triangle $P \subset \tilde{S} - \tilde{\lambda}$. Let $\varphi_{P_0P} \in \text{SL}_n(\mathbb{R})$ be the shearing map associated with P . We must verify that

$$\psi_{P_0g_P^\pm}(\mathcal{F}_\rho(x)) = \varphi_{P_0P}(\mathcal{F}_\rho(x)).$$

If $g = g_P^-$, then

$$\psi_{P_0g_P^-}(\mathcal{F}_\rho(x)) = \psi_{P_0P}(\mathcal{F}_\rho(x)) = \psi_{P_0P} \circ T_{g_P^-}^{\varepsilon(P_0,P)}(\mathcal{F}_\rho(x)) = \varphi_{P_0P}(\mathcal{F}_\rho(x))$$

since the flag $\mathcal{F}_\rho(x)$ is fixed by the linear map $T_{g_P^-}^{\varepsilon(P_0,P)}$. If $g = g_P^+$,

$$\psi_{P_0g_P^+}(\mathcal{F}_\rho(x)) = \psi_{P_0P} \circ T_{g_P^+}^{-\varepsilon(P_0,P)} \circ T_{g_P^-}^{\varepsilon(P_0,P)}(\mathcal{F}_\rho(x)) = \varphi_{P_0P}(\mathcal{F}_\rho(x))$$

since the flag $\mathcal{F}_\rho(x)$ is fixed by $T_{g_P^+}^{-\varepsilon(P_0,P)}$. As a result, $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$ is a well-defined map, which extends the previous map $\mathcal{F}'_\rho : \mathcal{V}_\lambda \rightarrow \text{Flag}(\mathbb{R}^n)$ of Lemma 27.

In particular, the restriction $\mathcal{F}'_{\rho|\mathcal{V}_\lambda} : \mathcal{V}_\lambda \rightarrow \text{Flag}(\mathbb{R}^n)$ is ρ' -equivariant. Let $x \in \partial_\infty \tilde{\lambda} - \mathcal{V}_\lambda$ be an endpoint of the leaf $g \subset \tilde{\lambda}$, and let $(g_n)_n \subset \tilde{\lambda}$ be a sequence of leaves converging to g , where each g_n bounds some triangle $R_n \subset \tilde{S} - \tilde{\lambda}$. Since $\lim_{n \rightarrow \infty} \psi_{P_{g_n}} = \psi_{Pg}$, and $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ is continuous,

$$\lim_{n \rightarrow \infty} \psi_{P_{g_n}}(\mathcal{F}_\rho(x_{g_n})) = \psi_{Pg}(\mathcal{F}_\rho(x))$$

where $(x_{g_n})_n \subset \partial_\infty \tilde{\lambda}$ is a sequence of endpoints of $(g_n)_n$ which converges to $x \in \partial_\infty \tilde{\lambda} - \mathcal{V}_\lambda$. The ρ' -equivariance property thus extends to the flag map $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$ by limiting process. \square

Now, let $\mathcal{F}_{\rho'} : \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ be the equivariant flag curve associated with the Anosov representation $\rho' = \Lambda^\varepsilon \rho$.

Theorem 30. *The restriction $\mathcal{F}_{\rho'}|_{\partial_\infty \tilde{\lambda}} : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$ coincides with the flag map $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$ of Lemma 29.*

Proof of Theorem 30. It is convenient to switch back to the Anosov section point of view. Indeed, it is the Anosov dynamics which makes everything work here.

We begin with a lemma. Given an Anosov representation $\rho : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, consider the flat bundle $T^1S \times_\rho \mathbb{R}^n \rightarrow T^1S$ of Remark 5. Let $(G_t)_{t \in \mathbb{R}}$ be the flow on $T^1S \times_\rho \mathbb{R}^n$ that lifts the geodesic flow $(g_t)_{t \in \mathbb{R}}$ on T^1S . The Anosov section $\sigma_\rho = (V_1, \dots, V_n)$ provides a line decomposition $V_1 \oplus \dots \oplus V_n$ of the bundle $T^1S \times_\rho \mathbb{R}^n \rightarrow T^1S$ with the property that each line sub-bundle $V_i \rightarrow T^1S$ is invariant under the action of the flow $(G_t)_{t \in \mathbb{R}}$. Finally, pick a metric $\|\cdot\|_u$ on $T^1S \times_\rho \mathbb{R}^n \rightarrow T^1S$.

Lemma 31. *For every $i > j$, for every $u \in T^1S$, for every vectors $X_i(u) \in V_i(u)$ and $X_j(u) \in V_j(u)$,*

$$\lim_{t \rightarrow +\infty} \frac{\|G_t X_j(u)\|_{g_t(u)}}{\|G_t X_i(u)\|_{g_t(u)}} = 0.$$

Proof. The lift $\|\cdot\|_{\tilde{u}}$ of $\|\cdot\|_u$ defines a $\pi_1(S)$ -invariant norm on \mathbb{R}^n . Let $\|\cdot\|_{\tilde{u}}$ be the induced norm on the vector space of linear endomorphisms $\text{End}(\mathbb{R}^n)$, namely, for every $\psi \in \text{End}(\mathbb{R}^n)$, for every $\tilde{u} \in T^1\tilde{S}$,

$$\|\psi\|_{\tilde{u}} = \sup_{X \in \mathbb{R}^n} \frac{\|\psi X\|_{\tilde{u}}}{\|X\|_{\tilde{u}}}.$$

By construction, $\|\cdot\|_{\tilde{u}}$ is $\pi_1(S)$ -invariant, and thus descends to a metric $\|\cdot\|_u$ on the flat bundle $T^1S \times_\rho \text{End}(\mathbb{R}^n) \rightarrow T^1S$. In particular, by restricting, $\|\cdot\|_u$ provides a metric on each line sub-bundle $V_i^* \otimes V_j \rightarrow T^1S$ of the bundle $T^1S \times_\rho \text{End}(\mathbb{R}^n) \rightarrow T^1S$; see §1.1.

Given $X_i(u) \in V_i(u)$ and $X_j(u) \in V_j(u)$, consider the vector $(X_i(u))^* \otimes X_j(u) \in V_i^* \otimes V_j(u)$. Recall that the action of the flow $(\bar{G}_t)_{t \in \mathbb{R}}$ on the line bundle $V_i^* \otimes V_j$ is contracting; see §1.1. Hence, for every $t > 0$,

$$\begin{aligned} \|G_t X_j(u)\|_{g_t(u)} &= \|[\bar{G}_t((X_i(\tilde{u}))^* \otimes X_j(\tilde{u}))](G_t X_i(\tilde{u}))\|_{g_t(\tilde{u})} \\ &\leq \|\bar{G}_t((X_i(u))^* \otimes X_j(u))\|_{g_t(u)} \|G_t X_i(u)\|_{g_t(u)} \\ &\leq A e^{-at} \|(X_i(u))^* \otimes X_j(u)\|_u \|G_t X_i(u)\|_{g_t(u)} \end{aligned}$$

for some $A \geq 0$ and $a > 0$, which proves the assertion. Note that in the above calculation, we use the fact that the connection on $T^1S \times_\rho \text{End}(\mathbb{R}^n) \rightarrow T^1S$ is flat in a crucial way. \square

Consider the associate flat M -bundle $T^1S \times_{\rho'} M \rightarrow T^1S$ of the Anosov representation $\rho' = \Lambda^\varepsilon \rho$. Identify the orientated geodesic lamination $\hat{\lambda}$ with its corresponding subset in T^1S ; note that $\hat{\lambda} \subset T^1S$ is a compact subset which is invariant under the action of the geodesic flow $(g_t)_{t \in \mathbb{R}}$. Let $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$ be the flag map of Lemma 29. Making use of \mathcal{F}'_ρ , we define a flat, continuous section $\sigma'_\rho = (W_1, \dots, W_n)$ over the geodesic lamination $\hat{\lambda}$ as follows; let $\tilde{\hat{\lambda}} \subset T^1\tilde{S}$ that lifts $\hat{\lambda} \subset T^1S$; for every $i = 1, \dots, n$, for every $\tilde{u} \in \tilde{\hat{\lambda}}$, set

$$\tilde{W}_i(\tilde{u}) = \mathcal{F}'_\rho^{(i)}(x_g^+) \cap \mathcal{F}'_\rho^{(n-i+1)}(x_g^-) \subset \mathbb{R}^n$$

where x_g^+ and $x_g^- \in \partial_\infty \tilde{S}$ are respectively the positive and the negative endpoints of the oriented geodesic $g \subset \tilde{\lambda}$ directed by the unit vector \tilde{u} . The ρ' -equivariance of the flag map $\mathcal{F}'_\rho : \partial_\infty \tilde{\lambda} \rightarrow \text{Flag}(\mathbb{R}^n)$ implies that the flat section $\tilde{\sigma}'_\rho = (\tilde{W}_1, \dots, \tilde{W}_n)$ (defined over $\tilde{\lambda}$) is ρ' -equivariant. In particular, it descends to a well-defined flat, continuous section $\sigma'_\rho = (W_1, \dots, W_n)$ over the geodesic lamination $\hat{\lambda} \subset T^1 \tilde{S}$ of the flat bundle $T^1 S \times_{\rho'} \mathbb{R}^n \rightarrow T^1 S$.

Now, let $\sigma_{\rho'} = (V'_1, \dots, V'_n)$ the Anosov section of the Anosov representation $\rho' = \Lambda^\varepsilon \rho$, and let $\sigma_{\rho'}|_{\hat{\lambda}} = (V'_1, \dots, V'_n)|_{\hat{\lambda}}$ be its restriction to $\hat{\lambda} \subset T^1 S$. We will show that the two flat sections $\sigma_{\rho'}|_{\hat{\lambda}} = (V'_1, \dots, V'_n)|_{\hat{\lambda}}$ and $\sigma'_\rho = (W_1, \dots, W_n)$ defined over $\hat{\lambda}$ coincide.

Let $T^1 S \times_{\rho'} \mathbb{R}^n \rightarrow T^1 S$ be the flat \mathbb{R}^n -bundle of Remark 5. Pick a metric $\|\cdot\|_u$ on $T^1 S \times_{\rho'} \mathbb{R}^n$. Given $u \in \hat{\lambda}$, pick a vector $Y_i(u) \in W_i(u)$ which lifts to $\tilde{Y}_i(\tilde{u}) \in \tilde{W}_i(\tilde{u}) \subset \mathbb{R}^n$. Let $\tilde{Y}_i(\tilde{u}) = \sum_{j=1}^n \tilde{X}'_j(\tilde{u})$ be its decomposition with respect to the line decomposition $\tilde{V}'_1(u) \oplus \dots \oplus \tilde{V}'_n(u) = \mathbb{R}^n$. We will prove that $\tilde{W}_i(\tilde{u}) = \tilde{V}'_i(\tilde{u})$.

Let $(t_k)_k \rightarrow +\infty$ such that $(g_{t_k} u)_k \subset \hat{\lambda}$ converges to $u_\infty \in \hat{\lambda}$ (such a $(t_k)_k$ exists since $\hat{\lambda}$ is compact); in particular, $(g_{t_k} \tilde{u})_k \subset \tilde{\lambda}$ projects to $(g_{t_k} u)_k \subset \hat{\lambda}$. Consider the vector $\frac{G_{t_k} \tilde{Y}_i(\tilde{u})}{\|G_{t_k} \tilde{X}'_{i_0}(\tilde{u})\|_{g_{t_k}(\tilde{u})}} \in \mathbb{R}^n$, where i_0 is the largest integer j such that the component $\tilde{X}'_j(\tilde{u}) \neq 0$. The connection on $T^1 S \times_{\rho'} \mathbb{R}^n$ being flat,

$$(2) \quad \frac{G_{t_k} \tilde{Y}_i(\tilde{u})}{\|G_{t_k} \tilde{X}'_{i_0}(\tilde{u})\|_{g_{t_k}(\tilde{u})}} = \frac{G_{t_k} \tilde{X}'_{i_0}(\tilde{u})}{\|G_{t_k} \tilde{X}'_{i_0}(\tilde{u})\|_{g_{t_k}(\tilde{u})}} + \sum_{j=1}^{i_0-1} \frac{G_{t_k} \tilde{X}'_j(\tilde{u})}{\|G_{t_k} \tilde{X}'_{i_0}(\tilde{u})\|_{g_{t_k}(\tilde{u})}}.$$

Likewise, $\sigma_{\rho'}|_{\hat{\lambda}} = (V'_1, \dots, V'_n)|_{\hat{\lambda}}$ being flat, for every k , for every $j = 1, \dots, i_0$,

$$(3) \quad \frac{G_{t_k} X'_j(u)}{\|G_{t_k} X'_{i_0}(u)\|_{g_{t_k}(u)}} \in V_j(g_{t_k}(u)).$$

By Lemma 31, and by continuity of the line sub-bundle $V'_j \rightarrow \hat{\lambda}$, for every $j = 1, \dots, i_0$, the sequence (3) converges to a vector in the fibre $V'_j(u_\infty)$; this vector is the zero vector for all $j \leq i_0 - 1$; and it is a unit vector for $j = i_0$; let $Z_0 \in V'_{i_0}(u_\infty)$ be this vector. It follows from (2) that

$$\lim_{k \rightarrow +\infty} \frac{G_{t_k} Y_i(u)}{\|G_{t_k} X'_{i_0}(u)\|_{g_{t_k}(u)}} = \lim_{k \rightarrow +\infty} \frac{G_{t_k} X'_{i_0}(u)}{\|G_{t_k} X'_{i_0}(u)\|_{g_{t_k}(u)}} = Z_{i_0} \in V'_{i_0}(u_\infty).$$

On the other hand, the section $\sigma'_\rho = (W_1, \dots, W_n)$ being flat and continuous, for every k , $\frac{G_{t_k} Y_i(u)}{\|G_{t_k} X'_{i_0}(u)\|_{g_{t_k}(u)}} \in W_i(g_{t_k}(u))$, and $Z_0 = \lim_{k \rightarrow +\infty} \frac{G_{t_k} Y_i(u)}{\|G_{t_k} X'_{i_0}(u)\|_{g_{t_k}(u)}} \in W_i(u_\infty)$. Since $\|Z_0\|_{u_\infty} = 1$, $Z_0 \neq 0$ in particular. Therefore, $W_i(u_\infty) = V'_{i_0}(u_\infty)$, and by flatness, the lines $\tilde{W}_i(\tilde{u})$ and $\tilde{V}'_{i_0}(\tilde{u}) \subset \mathbb{R}^n$ coincide.

The lines $\tilde{W}_1(\tilde{u}), \dots, \tilde{W}_n(\tilde{u})$ being linearly independent, it follows from the above discussion that, for every $u \in \hat{\lambda}$,

$$\sigma'_\rho(u) = (W_1(u), \dots, W_n(u)) = (V'_{i_1}(u), \dots, V'_{i_n}(u))$$

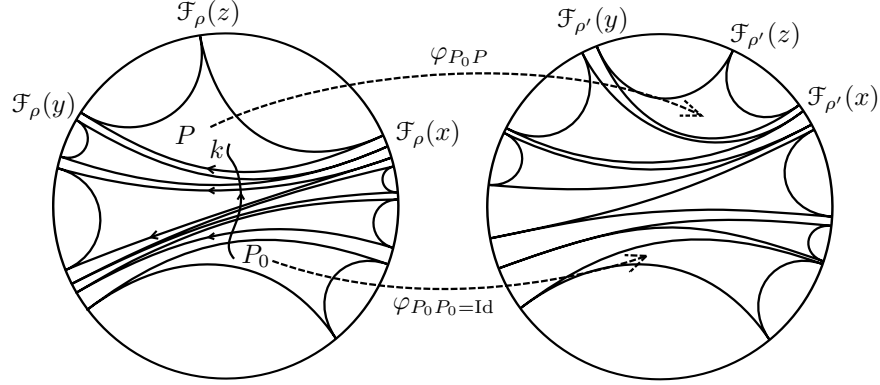


FIGURE 3. Shearing maps between ρ and its cataclysm deformation $\rho' = \Lambda^\varepsilon \rho$.

for some permutation $(i_1(u), \dots, i_n(u))$ of the set $\{1, \dots, n\}$ which depends on the point $u \in \hat{\lambda}$. Naturally, σ'_ρ and $\sigma_{\rho'}$ being flat, the n -tuple $u \mapsto (i_1(u), \dots, i_n(u))$ is a constant map along the leaves of $\hat{\lambda}$. In addition, λ being maximal, the geodesic lamination $\hat{\lambda}$ is connected. By continuity of the line bundles $W_i \rightarrow \hat{\lambda}$ and $V'_i \rightarrow \hat{\lambda}$, we conclude that the function $u \mapsto (i_1(u), \dots, i_n(u))$ is constant on $\hat{\lambda}$. Thus, for every $u \in \hat{\lambda}$,

$$\sigma'_\rho(u) = (W_1(u), \dots, W_n(u)) = (V'_{i_1}(u), \dots, V'_{i_n}(u))$$

for some permutation (i_1, \dots, i_n) of the set $\{1, \dots, n\}$.

Finally, consider $\tilde{\sigma}'_\rho = (\tilde{V}'_{i_1}, \dots, \tilde{V}'_{i_n})$ that lifts σ'_ρ . Let $\tilde{u}_0 \in \tilde{\hat{\lambda}}$ be a point along a leaf that projects to a geodesic leaf bounding the ideal triangle $P_0 \subset \tilde{S} - \tilde{\lambda}$. By construction of σ'_ρ , for every $j = 1, \dots, n$,

$$\tilde{V}'_{i_j}(\tilde{u}_0) = \varphi_{P_0 P_0}(\tilde{V}_j(\tilde{u}_0))$$

where $\varphi_{P_0 P_0} \in \mathrm{SL}_n(\mathbb{R})$ is the shearing map associated with the triangle P_0 , and where $\tilde{\sigma}_\rho = (\tilde{V}_1, \dots, \tilde{V}_n)$ lifts the Anosov section σ_ρ of the initial Anosov representation ρ . Since $\varphi_{P_0 P_0} = \mathrm{Id}$, $\tilde{V}'_{i_j}(\tilde{u}_0) = \tilde{V}_j(\tilde{u}_0)$ for every j , which implies that $i_j = j$. Hence, for every $u \in \hat{\lambda}$, $\sigma'_\rho(u) = \sigma_{\rho'}(u)$; equivalently, the flag maps \mathcal{F}'_ρ and $\mathcal{F}_{\rho'}|_{\partial_\infty \tilde{\lambda}}$ coincide on $\partial_\infty \tilde{\lambda} \subset \partial_\infty \tilde{S}$. This achieves the proof of Theorem 30. \square

Remark 32. Theorem 30 gives a simple, geometric description of a ε -cataclysm deformation $\rho' = \Lambda^\varepsilon \rho$: The ρ -equivariant flag curve $\mathcal{F}_\rho : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ is mapped onto the ρ' -equivariant flag curve $\mathcal{F}_{\rho'} : \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ via the equivariant family of shearing maps $\Lambda^\varepsilon = \{\varphi_{P_0 P}\}_{P \subset \tilde{S} - \tilde{\lambda}} \subset \mathrm{SL}_n(\mathbb{R})$. More precisely, if x, y and $z \in \partial_\infty \tilde{S}$ are the vertices of some ideal triangle $P \subset \tilde{S} - \tilde{\lambda}$, the shearing map $\varphi_{P_0 P}$ sends the flag triplet $\mathcal{F}_\rho(P) = (\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z))$ to the flag triplet $\mathcal{F}_{\rho'}(P) = (\mathcal{F}_{\rho'}(x), \mathcal{F}_{\rho'}(y), \mathcal{F}_{\rho'}(z))$; see Figure 3. In particular, a cataclysm can be viewed as a deformation of an Anosov representation ρ via a deformation of its associated flag curve \mathcal{F}_ρ .

Equivalently, in terms of Anosov sections, the cataclysm map Λ^ε sends the Anosov section $\sigma_\rho = (V_1, \dots, V_n)$ to the Anosov section $\sigma_{\rho'} = (V'_1, \dots, V'_n)$; see §1.2.

5. GEOMETRIC PROPERTIES OF CATACLYSMS

We now establish some geometric properties of cataclysms. In particular, the main goal of this section is to obtain a variation formula (Theorem 40) for the length functions ℓ_i^ρ [Dr₁] of an Anosov representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$.

5.1. The shear as a summation. Given a cataclysm deformation $\rho' = \Lambda^\varepsilon \rho$, we give a description of the shear $\varepsilon \in \mathbb{C}^{\mathrm{Twist}}(\widehat{\lambda})$ as a certain summation.

Let k be a transverse, simple, nonbacktracking, oriented arc to $\widehat{\lambda}$. Orient positively the leaves of $\widehat{\lambda}$ intersecting k for the transverse orientation determined by the oriented arc k . As in §2.1, for every component $d \subset k - \widehat{\lambda}$, g_d^+ and $g_d^- \subset \widehat{\lambda}$ are the two leaves passing by the positive and the negative endpoints of the oriented subarc $d \subset k - \widehat{\lambda}$. Let \widetilde{u}_d^+ and $\widetilde{u}_d^- \in T^1\widetilde{S}$ be the unit tangent vectors that direct the oriented leaves g_d^+ and $g_d^- \subset \widehat{\lambda}$, and are based at the positive and the negative endpoints of each oriented subarc $d \subset k - \widehat{\lambda}$, respectively. Finally, fix a triangle $P_0 \subset \widetilde{S} - \widehat{\lambda}$; let $R_d \subset \widetilde{S} - \widehat{\lambda}$ be the ideal triangle containing the subarc d ; and let $\varphi_d = \varphi_{P_0 R_d} \in \mathrm{SL}_n(\mathbb{R})$ be the associated shearing map.

Let $\sigma_\rho = (V_1, \dots, V_n)$ and $\sigma_{\rho'} = (V'_1, \dots, V'_n)$ be the Anosov sections of ρ and $\rho' = \Lambda^\varepsilon \rho$, respectively, that lift to $\widetilde{\sigma}_\rho = (\widetilde{V}_1, \dots, \widetilde{V}_n)$ and to $\widetilde{\sigma}_{\rho'} = (\widetilde{V}'_1, \dots, \widetilde{V}'_n)$, respectively. By Theorem 30 and Remark 32, for every subarc $d \subset k - \widehat{\lambda}$,

$$(4) \quad \varphi_d(\widetilde{V}_i(\widetilde{u}_d^\pm)) = \widetilde{V}'_i(\widetilde{u}_d^\pm).$$

Let $T^1S \times_\rho \bar{R}^n \rightarrow T^1S$ and $T^1S \times_{\rho'} \bar{R}^n \rightarrow T^1S$ be respectively the flat bundles of the Anosov representations ρ and $\rho' = \Lambda^\varepsilon \rho$ (see Remark 5), endowed with the metrics $\|\cdot\|_u$ and $\|\cdot\|'_u$, respectively. In particular, by restricting, for every $i = 1, \dots, n$, this provides a metric on each of the line sub-bundles $V_i \rightarrow T^1S$ and $V'_i \rightarrow T^1S$. Identify the oriented geodesic lamination $\widehat{\lambda}$ with its corresponding subset in T^1S . Pick a unit section $X_i : \widehat{\lambda} \rightarrow V_i$ (i.e. $\|X_i(u)\|_u = 1$ for every $u \in \widehat{\lambda}$), that lifts to $\widetilde{X}_i(\widetilde{u}) \in \widetilde{V}_i(\widetilde{u}) \subset \mathbb{R}^n$, $\widetilde{u} \in \widehat{\lambda}$ (such a section $\widetilde{X}_i : \widehat{\lambda} \rightarrow \widetilde{V}_i$ is not necessarily continuous). By (4), for every subarc $d \subset k - \widehat{\lambda}$,

$$\varphi_d \widetilde{X}_i(\widetilde{u}_d^\pm) \in \widetilde{V}'_i(\widetilde{u}_d^\pm) \subset \mathbb{R}^n.$$

For every $i = 1, \dots, n$, let $\delta_i^{\rho\rho'}(k)$ be the sum defined as

$$\begin{aligned} \delta_i^{\rho\rho'}(k) = \sum_{\substack{d \subset k - \widehat{\lambda} \\ d \neq d^\pm}} \log \frac{\left\| \varphi_d \widetilde{X}_i(\widetilde{u}_d^-) \right\|'_{\widetilde{u}_d^-}}{\left\| \varphi_d \widetilde{X}_i(\widetilde{u}_d^+) \right\|'_{\widetilde{u}_d^+}} & - \log \left\| \varphi_{d^-} \widetilde{X}_i(\widetilde{u}_{d^-}^+) \right\|'_{\widetilde{u}_{d^-}^+} \\ & + \log \left\| \varphi_{d^+} \widetilde{X}_i(\widetilde{u}_{d^+}^-) \right\|'_{\widetilde{u}_{d^+}^-} \end{aligned}$$

where the indexing d ranges over all the components in $k - \widehat{\lambda}$, and where d^+ and d^- are the two components containing respectively the positive and the negative

endpoints of the oriented arc k . Note that the value of the sum $\delta_i^{\rho\rho'}(k)$ is clearly independent of the choice of the lift $\tilde{X}_i(\tilde{u}) \in \tilde{V}_i(\tilde{u})$, $\tilde{u} \in \tilde{\lambda}$.

Lemma 33. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough, for every transverse, simple, non-backtracking, oriented arc k to $\tilde{\lambda}$, for every $i = 1, \dots, n$, the series $\delta_i^{\rho\rho'}(k)$ is absolutely convergent.*

Proof. Fix an arc k as above. Pick a metric $\|\cdot\|$ on \mathbb{R}^n . Since $k \cap \tilde{\lambda}$ is compact, the lifted metric $\|\cdot\|'_{|_{k \cap \tilde{\lambda}}}$ on the line bundle $\tilde{V}'_{i|_{k \cap \tilde{\lambda}}}$ is equivalent to the restriction of the metric $\|\cdot\|$ to $\tilde{V}'_{i|_{k \cap \tilde{\lambda}}} \subset \mathbb{R}^n$. In particular, to prove the absolute convergence of the series $\delta_i^{\rho\rho'}$, it is sufficient to show the convergence of the series

$$(5) \quad \sum_{d \subset k - \tilde{\lambda}} \left| \log \frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|} \right|.$$

To do so, we begin with finding an estimate for each term $\left| \log \frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|} \right|$ of this series.

By Theorem 6, the fibre $V_i(u)$ depends Hölder continuously on the point $u \in \hat{\lambda}$. Since the choice of the lift $\tilde{X}_i(\tilde{u})$ in (5) is irrelevant, we may assume the lift $\tilde{X}_i(\tilde{u}) \in \mathbb{R}^n$, $\tilde{u} \in \tilde{\lambda}$ to be locally Hölder continuous for the norm $\|\cdot\|$ of \mathbb{R}^n . By Lemma 13, for every subarc $d \subset k - \tilde{\lambda}$ whose divergence radius $r(d)$ (see §2.1) is large enough,

$$\|\tilde{X}_i(\tilde{u}_d^-) - \tilde{X}_i(\tilde{u}_d^+)\| = O(e^{-Kr(d)})$$

for some $K > 0$ (depending on k and ρ). By Corollary 20, $\varphi_d = \psi_{P_0 R_d} \circ T_{g_{R_d}}^{\varepsilon(P_0, R_d)}$, and thus

$$\|\varphi_d \tilde{X}_i(\tilde{u}_d^-) - \varphi_d \tilde{X}_i(\tilde{u}_d^+)\| \leq \|\psi_d\| \|T_{g_{R_d}}^{\varepsilon(P_0, R_d)}\| \|\tilde{X}_i(\tilde{u}_d^-) - \tilde{X}_i(\tilde{u}_d^+)\|.$$

The estimates in Corollary 20 and in Lemma 16 then show that, for every subarc $d \subset k - \tilde{\lambda}$ whose divergence radius $r(d)$ is large enough,

$$(6) \quad \|\varphi_d \tilde{X}_i(\tilde{u}_d^-) - \varphi_d \tilde{X}_i(\tilde{u}_d^+)\| = O(e^{C\|\varepsilon\|(r(d)+1)} e^{-Kr(d)})$$

for some $C \geq 0$ (depending on k and ρ).

We now determine a lower and upper bound for the term $\log \frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|}$. For every subarc $d \subset k - \tilde{\lambda}$,

$$(7) \quad \frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|} \leq 1 + \frac{1}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|} \|\varphi_d \tilde{X}_i(\tilde{u}_d^-) - \varphi_d \tilde{X}_i(\tilde{u}_d^+)\|.$$

Again, let us write $\varphi_d = \psi_{P_0 R_d} \circ T_{g_{R_d}}^{\varepsilon(P_0, R_d)}$. The estimate in Corollary 20 shows that the family $\{\psi_{P_0 R_d}\}_{d \subset k - \tilde{\lambda}} \subset \text{SL}_n(\mathbb{R})$ is bounded, and in addition, that it remains bounded away from $0 \in \text{Mat}_n(\mathbb{R})$, whenever $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ is small enough.

Therefore, for every $X \in \mathbb{R}^n$,

$$(8) \quad \|\varphi_d(X)\| \geq m \left\| T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)}(X) \right\|$$

for some $m > 0$ (depending on k and ρ). By combining estimates (6), (7) and (8),

$$\begin{aligned} \frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|} &\leq 1 + \frac{1}{m} \frac{1}{\left\| T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)}(\tilde{X}_i(\tilde{u}_d^-)) \right\|} \left\| \varphi_d \tilde{X}_i(\tilde{u}_d^-) - \varphi_d \tilde{X}_i(\tilde{u}_d^+) \right\| \\ &\leq 1 + \frac{1}{m} e^{-\varepsilon_i(P_0, R_d)} O(e^{C\|\varepsilon\|(r(d)+1)} e^{-Kr(d)}) \\ &\leq 1 + O(e^{2C\|\varepsilon\|(r(d)+1) - Kr(d)}) \end{aligned}$$

Hence, for every subarc $d \subset k - \tilde{\lambda}$ whose divergence radius $r(d)$ is large enough,

$$(9) \quad -O(e^{2C\|\varepsilon\|(r(d)+1) - Kr(d)}) \leq \log \frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|}.$$

Likewise, a similar calculation yields

$$\frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|} \leq 1 + \frac{1}{m} \frac{1}{\left\| T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)}(\tilde{X}_i(\tilde{u}_d^+)) \right\|} \left\| \varphi_d \tilde{X}_i(\tilde{u}_d^-) - \varphi_d \tilde{X}_i(\tilde{u}_d^+) \right\|.$$

Note that

$$\begin{aligned} \left\| T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)}(\tilde{X}_i(\tilde{u}_d^-)) - T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)}(\tilde{X}_i(\tilde{u}_d^+)) \right\| &= O\left(\left\| T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)} \right\| \left\| \tilde{X}_i(\tilde{u}_d^-) - \tilde{X}_i(\tilde{u}_d^+) \right\| \right) \\ &= O\left(e^{C\|\varepsilon\|(r(d)+1)} e^{-Kr(d)} \right). \end{aligned}$$

Hence

$$\begin{aligned} \left\| T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)}(\tilde{X}_i(\tilde{u}_d^+)) \right\| &\geq \left\| T_{g_{R_d}^-}^{\varepsilon(P_0, R_d)}(\tilde{X}_i(\tilde{u}_d^-)) \right\| - O\left(e^{C\|\varepsilon\|(r(d)+1)} e^{-Kr(d)} \right) \\ &\geq e^{-\varepsilon_i(P_0, R_d)} - O\left(e^{C\|\varepsilon\|(r(d)+1)} e^{-Kr(d)} \right) \\ &\geq e^{-C\|\varepsilon\|(r(d)+1)} - O\left(e^{C\|\varepsilon\|(r(d)+1)} e^{-Kr(d)} \right) \\ &\geq e^{-C\|\varepsilon\|(r(d)+1)} \left(1 - O\left(e^{2C\|\varepsilon\|(r(d)+1) - Kr(d)} \right) \right). \end{aligned}$$

Observe that, whenever $\|\varepsilon\| < K/2C$, for every subarc $d \subset k - \tilde{\lambda}$ whose divergence radius $r(d)$ is large enough, the right-hand side is positive. Therefore,

$$\begin{aligned} \frac{\|\varphi_d \tilde{X}_i(\tilde{u}_d^-)\|}{\|\varphi_d \tilde{X}_i(\tilde{u}_d^+)\|} &\leq 1 + \frac{1}{m} \frac{O(e^{C\|\varepsilon\|(r(d)+1) - Kr(d)})}{e^{-C\|\varepsilon\|(r(d)+1)} \left(1 - O\left(e^{2C\|\varepsilon\|(r(d)+1) - Kr(d)} \right) \right)} \\ &\leq 1 + \frac{1}{m} \frac{O(e^{2C\|\varepsilon\|(r(d)+1) - Kr(d)})}{\left(1 - O\left(e^{2C\|\varepsilon\|(r(d)+1) - Kr(d)} \right) \right)} \\ &\leq 1 + O(e^{2C\|\varepsilon\|(r(d)+1) - Kr(d)}). \end{aligned}$$

Hence, for every subarc $d \subset k - \tilde{\lambda}$ whose divergence radius $r(d)$ is large enough,

$$(10) \quad \log \frac{\left\| \varphi_d \tilde{X}_i(\tilde{u}_d^-) \right\|}{\left\| \varphi_d \tilde{X}_i(\tilde{u}_d^+) \right\|} \leq O(e^{2C\|\varepsilon\|(r(d)+1)-Kr(d)}).$$

The convergence of the series (5) then follows from estimates (9) and (10), and from an application of Lemma 12, whenever $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ is small enough.

Finally, note the following additivity property. Let k_1 and k_2 be two subarcs of k with disjoint interior such that $k = k_1 \cup k_2$, and assume that both series $\delta_i^{\rho\rho'}(k_1)$ and $\delta_i^{\rho\rho'}(k_2)$ are absolutely convergent. Then $\delta_i^{\rho\rho'}(k) = \delta_i^{\rho\rho'}(k_1) + \delta_i^{\rho\rho'}(k_2)$, which implies that $\delta_i^{\rho\rho'}(k)$ is also absolutely convergent. The same argument as in the proof of Lemma 25 then shows that we can find $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough so that, for every transverse, simple, nonbacktracking, oriented arc k to $\tilde{\lambda}$, the series $\delta_i^{\rho\rho'}(k)$ is absolutely convergent. \square

Remark 34. A consequence of the absolute convergence in Lemma 17 is that, for $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough, the series

$$\delta_i^{\rho\rho'}(k) = \sum_{\substack{d \subset k - \tilde{\lambda} \\ d \neq d^\pm}} \log \frac{\left\| \varphi_d \tilde{X}_i(\tilde{u}_d^-) \right\|'_{\tilde{u}_d^-}}{\left\| \varphi_d \tilde{X}_i(\tilde{u}_d^+) \right\|'_{\tilde{u}_d^+}} - \log \left\| \varphi_{d^-} \tilde{X}_i(\tilde{u}_{d^-}^+) \right\|'_{\tilde{u}_{d^-}^+} + \log \left\| \varphi_{d^+} \tilde{X}_i(\tilde{u}_{d^+}^-) \right\|'_{\tilde{u}_{d^+}^-}$$

is *commutatively* convergent.

Proposition 35. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough, for every transverse, simple, nonbacktracking arc k to $\tilde{\lambda}$, the n -tuple $\delta^{\rho\rho'}(k) = (\delta_1^{\rho\rho'}(k), \dots, \delta_n^{\rho\rho'}(k))$ is equal to the n -uplet $\varepsilon(k) = (\varepsilon_1(k), \dots, \varepsilon_n(k)) \in \mathbb{R}^n$.*

In the above statement, the transverse n -twisted cocycle $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ is regarded as a $\pi_1(S)$ -invariant transverse n -twisted cocycle for the lift $\tilde{\lambda}$.

Proof. Fix an arc k as above. By Lemma 33 and Remark 34, for every $i = 1, \dots, n$, whenever $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ is small enough,

$$\begin{aligned} \delta_i^{\rho\rho'}(k) = \lim_{r \rightarrow \infty} \sum_{\substack{d \subset k - \tilde{\lambda} \\ d \neq d^\pm \\ r(d) \leq r}} \log \frac{\left\| \varphi_d \tilde{X}_i(\tilde{u}_d^-) \right\|'_{\tilde{u}_d^-}}{\left\| \varphi_d \tilde{X}_i(\tilde{u}_d^+) \right\|'_{\tilde{u}_d^+}} & - \log \left\| \varphi_{d^-} \tilde{X}_i(\tilde{u}_{d^-}^+) \right\|'_{\tilde{u}_{d^-}^+} \\ & + \log \left\| \varphi_{d^+} \tilde{X}_i(\tilde{u}_{d^+}^-) \right\|'_{\tilde{u}_{d^+}^-} \end{aligned}$$

where $r(d)$ is the divergence radius of the subarc $d \subset k - \tilde{\lambda}$ (see §2.1). We wish to show that $\delta_i^{\rho\rho'}(k) = \varepsilon_i(k)$.

With the same notations as in §3, let P and $Q \subset \tilde{S} - \tilde{\lambda}$ be the two ideal triangles whose interiors are joined by the oriented transverse arc k to $\tilde{\lambda}$. Put $m_r = \text{Card}(\mathcal{P}_{PQ}^r)$. Index the elements of \mathcal{P}_{PQ}^r as $R_1^r, R_2^r, \dots, R_m^r$ so that the indexing j of R_j^r increases as one goes from P to Q , and for convenience, set $R_0^r = P$

and $R_{m+1}^r = Q$. Finally, let us set $d_j = k \cap R_j^r$. Then

$$\begin{aligned} \delta_i^{\rho\rho'}(k) = \lim_{r \rightarrow \infty} \sum_{j=1}^{m_r} \log \frac{\left\| \varphi_{d_j} \tilde{X}_i(\tilde{u}_j^-) \right\|'_{\tilde{u}_j^-}}{\left\| \varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+) \right\|'_{\tilde{u}_j^+}} & - \log \left\| \varphi_{d_0} \tilde{X}_i(\tilde{u}_0^+) \right\|'_{\tilde{u}_0^+} \\ & + \log \left\| \varphi_{d_{m+1}} \tilde{X}_i(\tilde{u}_{m+1}^-) \right\|'_{\tilde{u}_{m+1}^-}. \end{aligned}$$

We emphasize that in the above series, the endpoints $\tilde{u}_j^\pm \in k \cap \tilde{\lambda}$ of each oriented subarc $d_j \subset k - \tilde{\lambda}$ all depend on r . We simply mean to alleviate (a very little) the already heavy notations. By reordering the terms,

$$\delta_i^{\rho\rho'}(k) = \lim_{r \rightarrow \infty} \sum_{j=0}^{m_r} \log \frac{\left\| \varphi_{d_{j+1}} \tilde{X}_i(u_{j+1}^-) \right\|'_{u_{j+1}^-}}{\left\| \varphi_{d_j} \tilde{X}_i(u_j^+) \right\|'_{u_j^+}}.$$

Pick a metric $\| \cdot \|$ on \mathbb{R}^n . $k \cap \tilde{\lambda}$ being compact, the lifted metric $\| \cdot \|'_{|k \cap \tilde{\lambda}}$ on the line bundle $\tilde{V}'_{i|k \cap \tilde{\lambda}}$ is equivalent to the restriction of $\| \cdot \|$ to $\tilde{V}'_{i|k \cap \tilde{\lambda}} \subset \mathbb{R}^n$. Note that, by definition of \mathcal{P}_{PQ}^r , we have $r(d) > r$ for every subarc $d \subset k - \tilde{\lambda} \setminus \bigcup d_j$. Thus, by Lemma 13, for every $j = 1, \dots, m_r$,

$$\text{dist}_{T^1 S}(\tilde{u}_{j+1}^-, \tilde{u}_j^+) = O(e^{-Ar})$$

for some $A > 0$ (depending on k and ρ). The lifted metric $\| \cdot \|'_u$ on \mathbb{R}^n depending smoothly on $\tilde{u} \in T^1 \tilde{S}$, it follows from the above estimate that

$$\delta_i^{\rho\rho'}(k) = \lim_{r \rightarrow \infty} \sum_{j=0}^{m_r} \log \frac{\left\| \varphi_{d_{j+1}} \tilde{X}_i(\tilde{u}_{j+1}^-) \right\|'_{\tilde{u}_{j+1}^-}}{\left\| \varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+) \right\|'_{\tilde{u}_j^+}} = \lim_{r \rightarrow \infty} \sum_{j=0}^{m_r} \log \frac{\left\| \varphi_{d_{j+1}} \tilde{X}_i(\tilde{u}_{j+1}^-) \right\|}{\left\| \varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+) \right\|}.$$

We now focus attention on the series on the right-hand side and calculate its value. To do so, we begin with finding an estimate for each term of this series. By applying Corollary 20,

$$\begin{aligned} \varphi_{d_{j+1}} \tilde{X}_i(\tilde{u}_{j+1}^-) &= \varphi_{d_j} \psi_{R_j^r R_{j+1}^r} T_{g_{R_{j+1}^r}}^{\varepsilon(R_j^r, R_{j+1}^r)}(\tilde{X}_i(\tilde{u}_{j+1}^-)) \\ &= e^{\varepsilon_i(R_j^r, R_{j+1}^r)} \varphi_{d_j} \psi_{R_j^r R_{j+1}^r}(\tilde{X}_i(\tilde{u}_{j+1}^-)) \\ &= e^{\varepsilon_i(R_j^r, R_{j+1}^r)} \varphi_{d_j} \left(\tilde{X}_i(\tilde{u}_{j+1}^-) + \phi_j(\tilde{X}_i(\tilde{u}_{j+1}^-)) \right) \end{aligned}$$

where $\phi_j \in \mathrm{SL}_n(\mathbb{R})$ is a linear map such that $\|\phi_j\| = O\left(\sum_{R \in \mathcal{P}_{R_j^r, R_{j+1}^r}} e^{-Br(k \cap R)}\right)$ for some B (depending on k and ρ). Thus

$$\begin{aligned} \varphi_{d_{j+1}} \tilde{X}_i(\tilde{u}_{j+1}^-) &= e^{\varepsilon_i(R_j^r, R_{j+1}^r)} \varphi_{d_j} \left(\tilde{X}_i(\tilde{u}_j^+) + (\tilde{X}_i(\tilde{u}_{j+1}^-) - \tilde{X}_i(\tilde{u}_j^+)) + \phi_j(\tilde{X}_i(\tilde{u}_{j+1}^-)) \right) \\ &= e^{\varepsilon_i(R_j^r, R_{j+1}^r)} \varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+) + e^{\varepsilon_i(R_j^r, R_{j+1}^r)} \varphi_{d_j} (\tilde{X}_i(\tilde{u}_{j+1}^-) - \tilde{X}_i(\tilde{u}_j^+)) \\ &\quad + e^{\varepsilon_i(R_j^r, R_{j+1}^r)} \varphi_{d_j} \phi_j(\tilde{X}_i(\tilde{u}_{j+1}^-)). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\|\varphi_{d_{j+1}} \tilde{X}_i(\tilde{u}_{j+1}^-)\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|} &= e^{\varepsilon_i(R_j^r, R_{j+1}^r)} \left[1 + O\left(\frac{\|\varphi_{d_j} (\tilde{X}_i(\tilde{u}_{j+1}^-) - \tilde{X}_i(\tilde{u}_j^+))\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|}\right) \right. \\ &\quad \left. + O\left(\frac{\|\varphi_{d_j} \phi_j(\tilde{X}_i(\tilde{u}_{j+1}^-))\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|}\right) \right] \end{aligned}$$

which implies that

$$\begin{aligned} \log \frac{\|\varphi_{d_{j+1}} \tilde{X}_i(\tilde{u}_{j+1}^-)\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|} &= \varepsilon_i(R_j^r, R_{j+1}^r) + O\left(\frac{\|\varphi_{d_j} (\tilde{X}_i(\tilde{u}_{j+1}^-) - \tilde{X}_i(\tilde{u}_j^+))\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|}\right) \\ &\quad + O\left(\frac{\|\varphi_{d_j} \phi_j(\tilde{X}_i(\tilde{u}_{j+1}^-))\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|}\right). \end{aligned}$$

Similar arguments as in the proof of Lemma 33 show that, provided that $\|\varepsilon\| < K/2C$, for r large enough, for every $j = 1, \dots, m_r$,

$$\frac{\|\varphi_{d_j} (\tilde{X}_i(\tilde{u}_{j+1}^-) - \tilde{X}_i(\tilde{u}_j^+))\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|} = O(e^{2C\|\varepsilon\|(r+1)-Kr})$$

for some $K > 0$ and C (both depending on k and ρ). Likewise,

$$\frac{\|\varphi_{d_j} \phi_j(\tilde{X}_i(\tilde{u}_{j+1}^-))\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|} = O(e^{2C\|\varepsilon\|(r+1)-Kr}).$$

As a result,

$$\begin{aligned} \delta_i^{\rho\rho'}(k) &= \lim_{r \rightarrow \infty} \sum_{j=0}^{m_r} \log \frac{\|\varphi_{d_{j+1}} \tilde{X}_i(\tilde{u}_{j+1}^-)\|}{\|\varphi_{d_j} \tilde{X}_i(\tilde{u}_j^+)\|} = \lim_{r \rightarrow \infty} \sum_{j=0}^{m_r} \varepsilon_i(R_j^r, R_{j+1}^r) \\ &\quad + \lim_{r \rightarrow \infty} \sum_{j=0}^{m_r} O(e^{2C\|\varepsilon\|(r+1)-Ar}). \end{aligned}$$

Finally, observe that $\varepsilon_i(k) = \sum_{j=0}^{m_r} \varepsilon_i(R_j^r, R_{j+1}^r)$, and that, by Lemma 12, $m_r = \mathrm{Card}(\mathcal{P}_{PQ}^r) = O(r)$. We conclude that

$$\delta_i^{\rho\rho'}(k) = \varepsilon_i(k).$$

□

Corollary 36. *Let $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ be an Anosov representation, and let \mathcal{U}^ρ be some open neighborhood of $0 \in \mathcal{C}^{\mathrm{Twist}}(\widehat{\lambda})$ small enough. The cataclysm map*

$$\begin{aligned} \Lambda : \mathcal{U}^\rho &\rightarrow \mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}^{\mathrm{Anosov}}(S) \\ \varepsilon &\mapsto \Lambda^\varepsilon \rho \end{aligned}$$

is injective.

Proof. Let ε and $\varepsilon' \in \mathcal{C}^{\mathrm{Twist}}(\widehat{\lambda})$ be such that $\Lambda^\varepsilon \rho = \Lambda^{\varepsilon'} \rho$. Then $\varepsilon = \delta^{\rho\rho'} = \varepsilon'$ by Proposition 35. □

5.2. Length functions of an Anosov representation. In [Dr₁], we extend Thurston's length function [Th₁, Th₂, Bon₄, Bon₁] to an important class of Anosov representations known as *Hitchin representations* [La, Gui, FoGo]. In particular, to every Hitchin representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$, we associate n length functions $\ell_i^\rho : \mathcal{C}^H(S) \rightarrow \mathbb{R}$ defined on the space of *Hölder geodesic currents* $\mathcal{C}^H(S)$ [Bon₂]. The construction of the lengths ℓ_i^ρ extends to every Anosov representation and we begin with reviewing some of this construction.

Given an Anosov representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$, consider the flat bundle $T^1S \times_\rho \mathbb{R}^n \rightarrow T^1S$ of Remark 5. Let $(G_t)_{t \in \mathbb{R}}$ be the flow on $T^1S \times_\rho \mathbb{R}^n$ that lifts the geodesic flow $(g_t)_{t \in \mathbb{R}}$ on T^1S . The Anosov section $\sigma_\rho = (V_1, \dots, V_n)$ provides a line decomposition $V_1 \oplus \dots \oplus V_n$ of the bundle $T^1S \times_\rho \mathbb{R}^n \rightarrow T^1S$ with the property that each line sub-bundle $V_i \rightarrow T^1S$ is invariant under the action of the flow $(G_t)_{t \in \mathbb{R}}$. Finally, pick a metric $\|\cdot\|_u$ on $T^1S \times_\rho \mathbb{R}^n \rightarrow T^1S$.

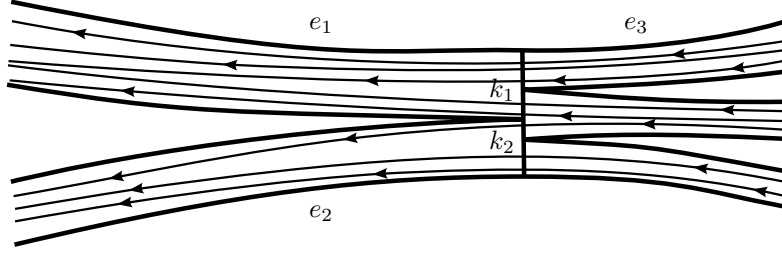
Let \mathcal{F} be the geodesic foliation of the unit tangent bundle T^1S . Let $u_0 \in T^1S$, and let $X_i(u_0) \in V_i(u_0)$ be a vector. For every $u \in T^1S$ lying on the same geodesic leaf as u_0 , set

$$\omega_i^\rho(u) = -\frac{d}{dt} \log \left\| G_t X_i(u_0) \right\|_{g_t(u_0)} \Big|_{t=t_u}$$

where $t_u \in \mathbb{R}$ is such that $u = g_{t_u}(u_0)$. The above expression defines a 1-form ω_i^ρ on T^1S along the leaves of the geodesic foliation \mathcal{F} . One easily verifies that the definition of ω_i^ρ is independent of the choices of $u_0 \in T^1S$ and $X_i(u_0) \in V_i(u_0)$, and thus only depends on the metric $\|\cdot\|_u$. In addition, because of the regularity of the line bundles $V_i \rightarrow T^1S$ (see Theorem 6), the 1-forms ω_i^ρ satisfy the following properties: they are smooth, closed along the leaves of the geodesic foliation \mathcal{F} , and are transversally Hölder continuous. We refer the reader to [Dr₁] for details.

Now, fix a maximal geodesic lamination $\lambda \subset S$ along with its orientation cover $\widehat{\lambda}$ as in §2.2. For the purpose of this paper, we will be only interested in length functions ℓ_i^ρ defined on the vector space $\mathcal{C}^H(\widehat{\lambda})$ of transverse cocycles for $\widehat{\lambda}$. In particular, we give an alternative definition of the lengths ℓ_i^ρ in the special case of $\mathcal{C}^H(\widehat{\lambda})$, which differs from the one in [Dr₁], but which better suits our needs and the context of this article.

Identify the oriented geodesic lamination $\widehat{\lambda}$ with its corresponding subset in T^1S ; note that $\widehat{\lambda} \subset T^1S$ is a closed subset which is the union of some leaves of the geodesic foliation \mathcal{F} . In particular, the geodesic lamination $\widehat{\lambda}$ inherits n 1-forms ω_i^ρ that are smooth, closed along its leaves, and transversally Hölder continuous. Let $\widehat{U} \supset \widehat{\lambda}$ be an open surface as in §2.2. We may assume without loss of generality that \widehat{U} is a train track [PeH, Bon₄] for the oriented lamination $\widehat{\lambda}$. Let e_1, \dots, e_m be the

FIGURE 4. An oriented geodesic lamination $\hat{\lambda}$ within a train track \hat{U} .

oriented edges of the train track \hat{U} ; and let $k_1, \dots, k_m \subset \hat{U}$ be the ingoing lids of each corresponding edge e_j ; see Figure 4. For every edge e_j , complete the partial foliation induced by $\hat{\lambda} \cap e_j$ in a full foliation of e_j . By integrating the 1-form ω_i^ρ along each oriented plaque in the edge e_j , and considering the negative endpoint of each oriented plaque, we define a function $h_j : k_j \rightarrow \mathbb{R}$ on the transverse arc k_j , which is Hölder continuous due to the regularity of ω_i^ρ . Let $\alpha \in \mathcal{C}^H(\hat{\lambda})$ be a transverse cocycle. By Theorem 18, α assigns on each transverse arc k_j a Hölder distribution α_{k_j} . The length $\ell_i^\rho(\alpha)$ of the transverse cocycle $\alpha \in \mathcal{C}^H(\hat{\lambda})$ is defined as

$$\ell_i^\rho(\alpha) = \sum_{j=1}^m \alpha_{k_j}(h_j) = \int_{\hat{\lambda}} \omega_i^\rho d\alpha$$

where $\alpha_{k_j}(h_j)$ is the value of the distribution α_{k_j} at the function h_j . One easily verifies that the value $\ell_i^\rho(\alpha)$ is independent of the choice of the train track $\hat{U} \supset \hat{\lambda}$. In addition, a homological argument shows that the value $\ell_i^\rho(\alpha)$ is independent of the metric $\|\cdot\|_u$ that we chose on the bundle $T^1S \times_\rho \mathbb{R}^n \rightarrow T^1S$, and of which we made use to define the 1-form ω_i^ρ . Finally, note that the length

$$\ell_i^\rho : \mathcal{C}^H(\hat{\lambda}) \rightarrow \mathbb{R}$$

is a linear function on the vector space of transverse cocycles $\mathcal{C}^H(\hat{\lambda})$.

5.3. 1-forms $\Delta_i^{\rho\rho'}$. Given an Anosov representation ρ , let $\rho' = \Lambda^\varepsilon \rho$ be a ε -cataclysm deformation along a maximal geodesic lamination $\lambda \subset S$ for some transverse n -twisted cocycle $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$ small enough. For every $i = 1, \dots, n$, set

$$\Delta_i^{\rho\rho'} = \omega_i^{\rho'} - \omega_i^\rho$$

where ω_i^ρ and $\omega_i^{\rho'}$ are the 1-forms as in §5.2. Therefore, $\Delta_i^{\rho\rho'}$ defines a 1-form that is smooth, closed along the leaves of the oriented geodesic lamination $\hat{\lambda}$, and is transversally Hölder continuous. We wish to relate the 1-form $\Delta_i^{\rho\rho'}$ to the shear $\varepsilon \in \mathcal{C}^{\text{Twist}}(\hat{\lambda})$. In particular, the main result of this section is Proposition 39.

Let $T^1S \times_\rho \bar{R}^n \rightarrow T^1S$ and $T^1S \times_{\rho'} \bar{R}^n \rightarrow T^1S$ be respectively the flat bundles of the Anosov representations ρ and $\rho' = \Lambda^\varepsilon \rho$, endowed with the metrics $\|\cdot\|_u$ and $\|\cdot\|'_u$, respectively. Let $V_i \rightarrow T^1S$ and $V'_i \rightarrow T^1S$ be the associated line bundles of ρ and ρ' .

Consider an open surface $\widehat{U} \supset \widehat{\lambda}$ as in §2.2. Recall that the complement $\widehat{U} - \widehat{\lambda}$ is made of one-holed ideal hexagons \widehat{P} , where each \widehat{P} is the lift of some (punctured) ideal triangle $P \subset S - \lambda$. Finally, identify the oriented geodesic lamination $\widehat{\lambda}$ with its corresponding subset in T^1S .

Let $\widehat{P} \subset \widehat{U} - \widehat{\lambda}$ be an one-holed hexagon, and let $h_1, \dots, h_6 \subset \widehat{\lambda}$ be the six (oriented) edges of \widehat{P} . Along the boundary $\partial\widehat{P} = h_1 \cup \dots \cup h_6$ of the one-holed hexagon \widehat{P} , we consider the function $F_{i,\partial\widehat{P}}$ defined as follows: let $\widetilde{\widehat{P}} \subset \widetilde{\widehat{U}} - \widetilde{\widehat{\lambda}}$ that lifts $\widehat{P} \subset \widehat{U} - \widehat{\lambda}$, with $\partial\widetilde{\widehat{P}} = \widetilde{h}_1 \cup \dots \cup \widetilde{h}_6$; for every $j = 1, \dots, 6$, for every $u \in h_j$, set

$$F_{i,\partial\widehat{P}|_{h_j}}(u) = -\log \left\| \varphi_{P_0\widetilde{\widehat{P}}} \widetilde{X}_i(\widetilde{u}) \right\|'_{\widetilde{u}}$$

where: $\widetilde{u} \in \widetilde{h}_j$ is a lift of u ; \widetilde{X}_i is the lift of some unit section X_i (for the metric $\| \cdot \|_u$) of the line bundle $V_i \rightarrow T^1S$; and $\varphi_{P_0\widetilde{\widehat{P}}}$ is the shearing map associated with the ideal triangle $\widetilde{\widehat{P}} \subset \widetilde{S} - \widetilde{\lambda}$ ($\widetilde{\widehat{P}}$ is the triangle such that the one-holed hexagon $\widetilde{\widehat{P}} \subset \widetilde{U} - \widetilde{\lambda}$ projects to $\widetilde{\widehat{P}}$; and $P_0 \subset \widetilde{S} - \widetilde{\lambda}$ is a triangle that we fix). A key step in proving Proposition 39 is the following observation.

Lemma 37. *For every $u \in \widehat{\lambda}$ that lies along the boundary $\partial\widehat{P}$ of the one-holed hexagon \widehat{P} ,*

$$\Delta_i^{\rho\rho'}(u) = d_u F_{i,\partial\widehat{P}}$$

where the differential is taken along $\partial\widehat{P}$.

In the above statement, $\| \cdot \|'_u$ is the metric chosen on the flat bundle $T^1S \times_{\rho'} \bar{\mathbb{R}}^n \rightarrow T^1S$ to define the 1-form $\omega_i^{\rho\rho'}$; see §5.2.

Proof. It follows from the equivariance property of the shearing map $\varphi_{P_0\widetilde{\widehat{P}}}$ that the function $F_{i,\partial\widehat{P}}$ is well defined. We must check that it is smooth.

Let $u \in \partial\widehat{P}$ that lies along the oriented edge $h_j \subset \widehat{\lambda}$, and let $\widetilde{\widehat{P}}, \widetilde{h}_j, \widetilde{u} \in \widetilde{h}_j$, and $\widetilde{\widehat{P}}$ as above. By Theorem 30 and Remark 32, for every $t \in \mathbb{R}$, $\varphi_{P_0\widetilde{\widehat{P}}} \widetilde{X}_i(g_t(\widetilde{u})) \in \widetilde{V}'_i(g_t(\widetilde{u})) \in \widetilde{V}_i(g_t(\widetilde{u}))$. Moreover, $X_i(u) \in V_i(u)$ being a unit section (for the metric $\| \cdot \|_u$), and the fibre $V_i(u)$ depending smoothly on $u \in T^1S$ along the leaves of the geodesic foliation \mathcal{F} , one easily verifies that the function

$$t \mapsto \log \left\| \varphi_{P_0\widetilde{\widehat{P}}} \widetilde{X}_i(g_t(\widetilde{u})) \right\|'_{g_t(\widetilde{u})}$$

is differentiable, which implies that $F_{i,\partial\widehat{P}}$ is smooth. Therefore, for every $u \in \widehat{P}$,

$$d_u F_{i,\partial\widehat{P}} = d_{\widetilde{u}} \log \left\| \varphi_{P_0\widetilde{\widehat{P}}} \widetilde{X}_i(\widetilde{u}) \right\|'_{\widetilde{u}} = \frac{d}{dt} \left(\log \left\| \varphi_{P_0\widetilde{\widehat{P}}} \widetilde{X}_i(g_t(\widetilde{u})) \right\|'_{g_t(\widetilde{u})} \right) dt \Big|_{t=0}$$

defines a smooth, closed 1-form along $\partial\widehat{P} \subset \widehat{\lambda}$.

By definition of the 1-forms ω_i^ρ and $\omega_i^{\rho\rho'}$ (see §5.2), for every $u \in \widehat{\lambda}$,

$$\Delta_i^{\rho\rho'}(u) = (\omega_i^{\rho\rho'} - \omega_i^\rho)(u) = \frac{d}{dt} \left(\log \left\| \widetilde{G}_t \widetilde{X}_i(\widetilde{u}) \right\|_{g_t(\widetilde{u})} - \log \left\| \widetilde{G}'_t \widetilde{X}'_i(\widetilde{u}) \right\|'_{g_t(\widetilde{u})} \right) dt \Big|_{t=0}$$

where $(G_t)_{t \in \mathbb{R}}$ and $(G'_t)_{t \in \mathbb{R}}$ are the flows on the flat bundles $T^1S \times_{\rho} \bar{\mathbb{R}}^n \rightarrow T^1S$ and $T^1S \times_{\rho'} \bar{\mathbb{R}}^n \rightarrow T^1S$, respectively; see Remark 5.

Let $\tilde{X}'_i(\tilde{u})$ be the lift of some unit section $X'_i(u)$ (for the metric $\|\cdot\|'_u$) of the line bundle $V'_i \rightarrow T^1S$. Since, for every $\tilde{u} \in \tilde{\lambda}$, $\varphi_{P_0\tilde{P}}(\tilde{V}_i(\tilde{u})) = \tilde{V}'_i(\tilde{u})$, we have $\varphi_{P_0\tilde{P}}\tilde{X}_i(\tilde{u}) = \mu_{\tilde{u}}\tilde{X}'_i(\tilde{u})$ for some $\mu_{\tilde{u}} \in \mathbb{R}$. In addition, because of the flat connections on $T^1S \times_{\rho} \tilde{R}^n$ and $T^1S \times_{\rho'} \tilde{R}^n$, and since the shearing map $\varphi_{P_0\tilde{P}}$ is a linear map, for every $t \in \mathbb{R}$,

$$\tilde{G}'_t\tilde{X}'_i(\tilde{u}) = \mu_{\tilde{u}}\varphi_{P_0\tilde{P}}(\tilde{G}_t\tilde{X}_i(\tilde{u})).$$

As a result, for every $t \in \mathbb{R}$, for every $\tilde{u} \in \partial\tilde{P}$,

$$\begin{aligned} \log \left\| \tilde{G}_t\tilde{X}_i(\tilde{u}) \right\|_{g_t(\tilde{u})} - \log \left\| \tilde{G}'_t\tilde{X}'_i(\tilde{u}) \right\|'_{g_t(\tilde{u})} &= -\log \frac{\left\| \tilde{G}'_t\tilde{X}'_i(\tilde{u}) \right\|'_{g_t(\tilde{u})}}{\left\| \tilde{G}_t\tilde{X}_i(\tilde{u}) \right\|_{g_t(\tilde{u})}} \\ &= -\log \frac{|\mu_{\tilde{u}}| \left\| \varphi_{P_0\tilde{P}}(\tilde{G}_t\tilde{X}_i(\tilde{u})) \right\|'_{g_t(\tilde{u})}}{\left\| \tilde{G}_t\tilde{X}_i(\tilde{u}) \right\|_{g_t(\tilde{u})}} \\ &= -\log \left\| \varphi_{P_0\tilde{P}} \left(\frac{\tilde{G}_t\tilde{X}_i(\tilde{u})}{\left\| \tilde{G}_t\tilde{X}_i(\tilde{u}) \right\|_{g_t(\tilde{u})}} \right) \right\|'_{g_t(\tilde{u})} \\ &\quad - \log |\mu_{\tilde{u}}| \\ &= -\log \left\| \varphi_{P_0\tilde{P}}\tilde{X}_i(g_t(\tilde{u})) \right\|'_{g_t(\tilde{u})} - \log |\mu_{\tilde{u}}|. \end{aligned}$$

Note that the very last step makes use of the fact that $X_i(u)$ is a unit section (for the metric $\|\cdot\|_u$) of the line bundle $V_i \rightarrow T^1S$.

We thus conclude that, for every $u \in \hat{\lambda}$ that that lies along the boundary $\partial\hat{P}$ of some one-holed hexagon \hat{P} ,

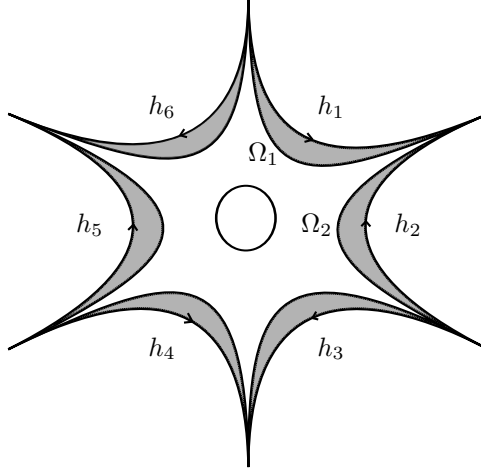
$$\begin{aligned} \Delta_i^{\rho\rho'}(u) &= \frac{d}{dt} \left(-\log \left\| \varphi_{P_0\tilde{P}}\tilde{X}_i(g_t(\tilde{u})) \right\|'_{g_t(\tilde{u})} - \log |\mu_{\tilde{u}}| \right)_{|t=0} dt \\ &= -d_{\tilde{u}} \log \left\| \varphi_{P_0\tilde{P}}\tilde{X}_i(\tilde{u}) \right\|'_{\tilde{u}}. \end{aligned}$$

□

Lemma 38. *The 1-form $\Delta_i^{\rho\rho'}$ (defined along the leaves of $\hat{\lambda}$) extends a Hölder continuous, closed 1-form defined on the open surface \hat{U} .*

Note that a Hölder continuous 1-form ω on \hat{U} is said to be *closed* if its path integral $\int_{\gamma} \omega$ along any path $\gamma \subset \hat{U}$ depends locally only on the endpoints of γ (in other words, the endpoints of γ being frozen, a small perturbation of γ does not change the value of $\int_{\gamma} \omega$).

Proof. Let $\hat{P} \subset \hat{U} - \hat{\lambda}$ be an one-holed hexagon, and let $h_1, \dots, h_6 \subset \hat{\lambda}$ be the six (oriented) edges of \hat{P} . Consider the function $F_{i,\partial\hat{P}}$ of Lemma 37. Let $\Omega_1, \dots, \Omega_6 \subset \hat{P}$ be open neighborhoods of the six edges h_1, \dots, h_6 , respectively, as shown

FIGURE 5. A one-holed hexagon \widehat{P} in $\widehat{U} - \widehat{\lambda}$.

on Figure 5; we choose the Ω_j so that their closures $\bar{\Omega}_i$ are pairwise disjoint. For every $j = 1, \dots, 6$, let $\theta_j : \widehat{P} \rightarrow \mathbb{R}$ be a bump function, which is identically equal to 1 near h_j , and which vanishes outside of Ω_j . Then $F_{i,\widehat{P}} = \sum \theta_j F_{i,\partial\widehat{P}|_{h_j}}$ is a smooth function on the interior of \widehat{P} that extends the previous function $F_{i,\partial\widehat{P}}$ defined along the boundary $\partial\widehat{P}$. Its differential $dF_{i,\widehat{P}}$ provides a smooth, exact 1-form on the interior of \widehat{P} , that extends the previous $dF_{i,\partial\widehat{P}}$ defined along $\partial\widehat{P}$. Therefore, by Lemma 37

$$dF_{i,\widehat{P}}|_{\partial\widehat{P}} = \Delta_i^{\rho\rho'}.$$

As a result, $\Delta_i^{\rho\rho'}$ (defined along $\widehat{\lambda}$) extends to a 1-form defined on \widehat{U} that is smooth, exact on the interior of each one-holed hexagon $\widehat{P} \subset \widehat{U} - \widehat{\lambda}$. Moreover, it follows from the construction, and the Hölder regularity of the 1-form $\Delta_i^{\rho\rho'}$ along the leaves of $\widehat{\lambda}$, that the extension $\Delta_i^{\rho\rho'}$ is Hölder continuous on \widehat{U} . In particular, for any path $\gamma \subset \widehat{U}$, the path integral $\int_\gamma \Delta_i^{\rho\rho'}$ is thus well defined. Besides, because of the exactness of $\Delta_i^{\rho\rho'}$ on the interior of each one-holed hexagon in $\widehat{U} - \widehat{\lambda}$, one easily verifies that the integral $\int_\gamma \Delta_i^{\rho\rho'}$ depends locally only on the endpoints of γ , which proves that the Hölder continuous 1-form $\Delta_i^{\rho\rho'}$ is closed. \square

Consider the Hölder continuous, closed 1-form $\Delta_i^{\rho\rho'}$ of Lemma 38, that is defined on $\widehat{U} \supset \widehat{\lambda}$.

Proposition 39. *For $\varepsilon \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ small enough, for every transverse, simple, nonbacktracking, oriented arc $k \subset \widehat{U}$ to $\widehat{\lambda}$,*

$$\varepsilon_i(k) = \int_k \Delta_i^{\rho\rho'} - F_{i,\widehat{P}^+}(u_k^+) + F_{i,\widehat{P}^-}(u_k^-).$$

where: u_k^+ and u_k^- are respectively the positive and the negative endpoints of the oriented arc k ; and \widehat{P}^+ and \widehat{P}^- are the one-holed hexagons containing the endpoints u_k^+ and u_k^- , respectively.

Proof. Let $k \subset \widehat{U}$ be a transverse, oriented arc to $\widehat{\lambda}$ as above, that lifts to arc $\widetilde{k} \subset \widetilde{U}$ transverse to $\widetilde{\lambda}$. Then

$$\int_k \Delta_i^{\rho\rho'} = \int_{\widetilde{k}} \Delta_i^{\rho\rho'} = \sum_{\widetilde{d} \subset \widetilde{k} - \widetilde{\lambda}} \int_{\widetilde{d}} \Delta_i^{\rho\rho'}$$

where the indexing \widetilde{d} ranges over the set of components of $\widetilde{k} - \widetilde{\lambda}$.

Recall that the 1-form $\Delta_i^{\rho\rho'}$ is exact in the interior of each one-holed hexagon in $\widehat{U} - \widehat{\lambda}$, and that, for every (oriented) subarc $\widetilde{d} \subset \widetilde{k} - \widetilde{\lambda}$

$$\int_{\widetilde{d}} \Delta_i^{\rho\rho'} = F_{i,\widetilde{d}}(\widetilde{u}_d^+) - F_{i,\widetilde{d}}(\widetilde{u}_d^-)$$

where: $F_{i,\widetilde{d}}$ is the function of Lemma 38 defined on the interior of the one-holed hexagon in $\widetilde{U} - \widetilde{\lambda}$ that contains the subarc $\widetilde{d} \subset \widetilde{k} - \widetilde{\lambda}$; and \widetilde{u}_d^+ and \widetilde{u}_d^- are respectively the positive and the negative endpoints of $\widetilde{d} \subset \widetilde{k} - \widetilde{\lambda}$. In particular, for every subarc $\widetilde{d} \subset \widetilde{k} - \widetilde{\lambda}$ which does not contain any of the endpoints u_k^+ and u_k^- of k ,

$$\int_{\widetilde{d}} \Delta_i^{\rho\rho'} = -\log \left\| \varphi_{\widetilde{d}} \widetilde{X}_i(\widetilde{u}_d^+) \right\|'_{\widetilde{u}_d^+} + \log \left\| \varphi_{\widetilde{d}} \widetilde{X}_i(\widetilde{u}_d^-) \right\|'_{\widetilde{u}_d^-}$$

where $\varphi_{\widetilde{d}} \in \mathrm{SL}_n(\mathbb{R})$ is the shearing map associated with the ideal triangle of $\widetilde{S} - \widetilde{\lambda}$ containing the subarc \widetilde{d} . In addition,

$$\int_{\widetilde{d}^+} \Delta_i^{\rho\rho'} = F_{i,\widetilde{d}^+}(\widetilde{u}_{d^+}^+) + \log \left\| \varphi_{\widetilde{d}^+} \widetilde{X}_i(\widetilde{u}_{d^+}^-) \right\|'_{\widetilde{u}_{d^+}^-}$$

and

$$\int_{\widetilde{d}^-} \Delta_i^{\rho\rho'} = -\log \left\| \varphi_{\widetilde{d}^-} \widetilde{X}_i(\widetilde{u}_{d^-}^+) \right\|'_{\widetilde{u}_{d^-}^+} - F_{i,\widetilde{d}^-}(\widetilde{u}_{d^-}^-)$$

where \widetilde{d}^+ and \widetilde{d}^- are the (oriented) subarcs containing the positive and the negative endpoints u_k^+ and u_k^- . As a result,

$$\begin{aligned} \int_k \Delta_i^{\rho\rho'} &= \sum_{\substack{\widetilde{d} \subset \widetilde{k} - \widetilde{\lambda} \\ \widetilde{d} \neq \widetilde{d}^\pm}} \log \frac{\left\| \varphi_{\widetilde{d}} \widetilde{X}_i(\widetilde{u}_d^-) \right\|'_{\widetilde{u}_d^-}}{\left\| \varphi_{\widetilde{d}} \widetilde{X}_i(\widetilde{u}_d^+) \right\|'_{\widetilde{u}_d^+}} + F_{i,\widetilde{d}^+}(u_k^+) + \log \left\| \varphi_{\widetilde{d}^+} \widetilde{X}_i(\widetilde{u}_{d^+}^-) \right\|'_{\widetilde{u}_{d^+}^-} \\ &\quad - \log \left\| \varphi_{\widetilde{d}^-} \widetilde{X}_i(\widetilde{u}_{d^-}^+) \right\|'_{\widetilde{u}_{d^-}^+} - F_{i,\widetilde{d}^-}(u_k^-) \end{aligned}$$

since $\widetilde{u}_{d^+}^+ = u_k^+$ and $\widetilde{u}_{d^-}^- = u_k^-$. The result then follows from Proposition 35, provided that $\varepsilon \in \mathcal{C}^{\mathrm{Twist}}(\widehat{\lambda})$ is small enough. \square

5.4. Thurston's intersection number. The vector space $\mathcal{C}^H(\widehat{\lambda})$ of transverse cocycles for $\widehat{\lambda}$ admits a natural symplectic form $\tau : \mathcal{C}^H(\widehat{\lambda}) \times \mathcal{C}^H(\widehat{\lambda}) \rightarrow \mathbb{R}$ known as *Thurston's intersection number* [Th2, Bon3, Bon4]. We review how this pairing is defined.

Consider an open surface $\widehat{U} \supset \widehat{\lambda}$ as in §2.2. Let $k_1, \dots, k_m \subset \widehat{U}$ be a finite family of disjoint transverse arcs to the geodesic lamination $\widehat{\lambda}$ such that every leaf intersects at least one k_j . Thus, $\widehat{\lambda} - \bigcup k_j$ is made of oriented arcs which can be regrouped into finitely many parallel classes; two oriented arcs belong to the same parallel class if their positive (negative resp.) endpoints lie in the same arcs $k_{j'}$ (k_j resp.). Collapse each k_j to a point u_j , and each parallel class to an oriented edge joining u_j to $u_{j'}$. We obtain an oriented graph \mathcal{G}_α with weights assigned on each of the edges as follows: if k is a transverse arc intersecting exactly all the leaves of a given parallel class, the corresponding edge of \mathcal{G}_α is assigned the weight $\alpha(k)$.

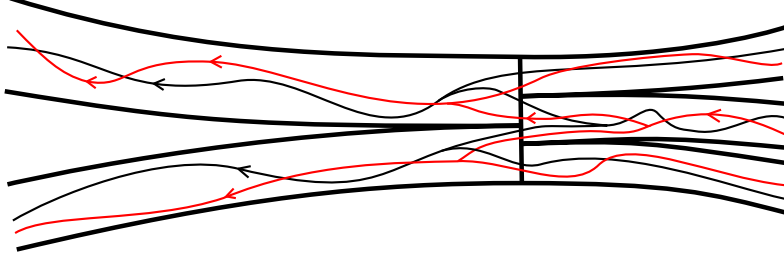


FIGURE 6. Thurston's intersection number of two transverse cocycles.

Given α and $\beta \in \mathcal{C}^H(\widehat{\lambda})$, the pairing $\tau(\alpha, \beta)$ is the self-intersection number between the two weighted oriented graphs \mathcal{G}_α and \mathcal{G}_β defined as follows: apply to the weighted graph \mathcal{G}_β a small perturbation so that the so-obtained weighted graph \mathcal{G}'_β is in transverse position to \mathcal{G}_α as on Figure 6; then assign to each intersection point of two edges the product of the corresponding weights, multiplied by $+1$ or -1 depending on whether the angle between the two oriented edges is positively or negatively oriented; and take the sum of all of these numbers. It is easy to verify that the resulting number does not depend on the choice of the graphs \mathcal{G}_α and \mathcal{G}'_β , and thus that the pairing $\tau(\alpha, \beta)$ is well defined. Note that the intersection number $\tau(\alpha, \beta)$ can be related to the classical self-intersection pairing in homology. Indeed, it follows from the additivity property of the transverse cocycle α that the oriented weighted graph \mathcal{G}_α is a 1-cycle in \widehat{U} . Hence $\alpha \in \mathcal{C}^H(\widehat{\lambda})$ defines a homology class $[\alpha] \in H_1(\widehat{U})$. In particular, Thurston's intersection number on $\mathcal{C}^H(\widehat{\lambda})$ coincides with the classical homology intersection pairing defined on $H_1(\widehat{U})$ (up to a nonzero scalar multiplication).

5.5. Variation of the length functions. We now describe the behavior of the length functions ℓ_i^ρ of §5.2 under cataclysm deformations.

Fix a maximal lamination $\lambda \subset S$ with orientation cover $\widehat{\lambda}$. Let ρ be an Anosov representation, and let $\rho' = \Lambda^\varepsilon \rho$ be a cataclysm deformation for some transverse n -twisted cocycle $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}^{\text{Twist}}(\widehat{\lambda})$ small enough. Let $\ell_i^\rho : \mathcal{C}^H(\widehat{\lambda}) \rightarrow \mathbb{R}$

and $\ell_i^{\rho'} : \mathcal{C}^H(\widehat{\lambda}) \rightarrow \mathbb{R}$ be respectively the length functions associated with ρ and ρ' ; see §5.2.

Theorem 40. *For every transverse cocycle $\alpha \in \mathcal{C}^H(\widehat{\lambda})$,*

$$\ell_i^{\rho'}(\alpha) = \ell_i^\rho(\alpha) + \tau(\alpha, \varepsilon_i)$$

where $\tau : \mathcal{C}^H(\widehat{\lambda}) \times \mathcal{C}^H(\widehat{\lambda}) \rightarrow \mathbb{R}$ is Thurston's intersection number.

Proof of Theorem 40. Let $\alpha \in \mathcal{C}^H(\widehat{\lambda})$. Then

$$\begin{aligned} \ell_i^{\rho'}(\alpha) - \ell_i^\rho(\alpha) &= \int_{\widehat{\lambda}} \omega_i^{\rho'} d\alpha - \int_{\widehat{\lambda}} \omega_i^\rho d\alpha \\ &= \int_{\widehat{\lambda}} \Delta_i^{\rho\rho'} d\alpha. \end{aligned}$$

Let $\mathcal{G}_\alpha = \sum a_p c_p$ be a weighted graph as in §5.4. Applying a small deformation, we can arrange that each 1-simplex c_j is simple and in transverse position to $\widehat{\lambda}$ and nonbacktracking. Finally, let \widehat{U} an open surface containing $\widehat{\lambda}$ as in §2.2, and let us extend the 1-form $\Delta_i^{\rho\rho'}$ defined along the leaves of $\widehat{\lambda}$ to \widehat{U} as in §5.3.

Lemma 41. *For every $\alpha \in \mathcal{C}^H(\widehat{\lambda})$,*

$$\int_{\widehat{\lambda}} \Delta_i^{\rho\rho'} d\alpha = \int_{\mathcal{G}_\alpha} \Delta_i^{\rho\rho'} d\alpha.$$

Proof. As in §5.4, pick a finite family of transverse, simple, nonbacktracking arcs $k_1, \dots, k_m \subset \widehat{U}$ to $\widehat{\lambda}$, so that $\widehat{\lambda} - \bigcup_j k_j$ consists of oriented arcs of finite length. Given two transverse arcs $k_{j'}$ and k_j , consider the set of oriented arcs in $\widehat{\lambda} - \bigcup_j k_j$ whose all positive endpoints lie in $k_{j'}$, and all negative endpoints lie in k_j . Let c_p be a 1-simplex intersecting k_j and $k_{j'}$. By subdividing the chain $\sum a_p c_p$ into a sum of smaller simplexes if necessary, we may assume without loss of generality that the positive and negative endpoints of c_p lie in $k_{j'}$ and k_j , respectively. Similarly, by subdividing each transverse arc k_j into smaller transverse subarcs, we may assume that two oriented arcs in $\widehat{\lambda} - \bigcup_j k_j$ whose negative endpoints lie in the same arc k_j also have their positive endpoints lying in the same arc $k_{j'}$. Recall that the length $\ell_i^\rho(\alpha)$ of the transverse cocycle $\alpha \in \mathcal{C}^H(\widehat{\lambda})$ is defined as

$$\ell_i^\rho(\alpha) = \sum_{j=1}^m \alpha_{k_j}(h_j)$$

where $\alpha_{k_j}(h_j)$ is the value of the transverse Hölder distribution α_{k_j} ; see §2.2 and §5.2.

For every $j = 1, \dots, m$, for every $u \in k_j$, consider the difference

$$s_j(u) = h_j(u) - \int_{c_p} \Delta_i^{\rho\rho'} = \int_{\text{arc}_u} \Delta_i^{\rho\rho'} - \int_{c_p} \Delta_i^{\rho\rho'}.$$

where arc_u is the oriented arc in $\widehat{\lambda} - \bigcup_j k_j$ whose negative endpoint is $u \in k_j$. Assuming the 1-simplex c_p to be small enough, it is contained in a simply connected open subset of \widehat{U} . The 1-form $\Delta_i^{\rho\rho'}$ being smooth, closed on this open subset, it is

thus exact. Therefore,

$$s_j(u) = \int_{\text{arc}_u} \Delta_i^{\rho\rho'} - \int_{c_p} \Delta_i^{\rho\rho'} = \int_{k_j u \rightarrow c_p(0)} \Delta_i^{\rho\rho'} - \int_{k_{j'} u' \rightarrow c_p(1)} \Delta_i^{\rho\rho'}$$

where: $c_p(0)$ and $c_p(1)$ are respectively the negative and the positive endpoints of the 1-simplex c_p ; $k_j u \rightarrow c_p(0)$ is the oriented subarc contained in the transverse arc k_j joining u to $c_p(0)$; and $k_{j'} u' \rightarrow c_p(1)$ is the oriented subarc contained in the transverse arc $k_{j'}$ joining u' to $c_p(1)$. Note that the function $s_j : k_j \rightarrow \mathbb{R}$ is Hölder continuous. As a result,

$$\sum_{j=1}^m \alpha_{k_j} (s_j) = \sum_{j=1}^m \alpha_{k_j} \left(\int_{k_j u \rightarrow c_p(0)} \Delta_i^{\rho\rho'} \right) - \sum_{j=1}^m \alpha_{k_j} \left(\int_{k_{j'} u' \rightarrow c_p(1)} \Delta_i^{\rho\rho'} \right) = 0.$$

We then conclude that

$$\ell_i^{\rho'}(\alpha) - \ell_i^\rho(\alpha) = \int_{\hat{\lambda}} \Delta_i^{\rho\rho'} d\alpha = \sum_{j=1}^m \alpha_{k_j} (h_j) = \int_{\sum a_p c_p} \Delta_i^{\rho\rho'} d\alpha$$

which proves the requested result. \square

Applying Lemma 40,

$$\begin{aligned} \ell_i^{\rho'}(\alpha) - \ell_i^\rho(\alpha) &= \int_{\hat{\lambda}} \Delta_i^{\rho\rho'} d\alpha \\ &= \int_{\sum a_p c_p} \Delta_i^{\rho\rho'} d\alpha \\ &= \sum a_p \int_{c_p} \Delta_i^{\rho\rho'} d\alpha \\ &= \sum [\pm 1]_p a_p \varepsilon_i(c_p) + \sum a_p [F_i(c_p(1)) - F_i(c_p(0))] \end{aligned}$$

where the latter step follows from an application of Proposition 39 (here, F_i denotes invariably any of the functions F_{i, \hat{p}_+} as in the proof of Lemma 38, depending on the one-hexagons the endpoints $c_p(1)$ and $c_p(0)$ belong to. Since $\mathcal{G}_\alpha = \sum a_p c_p$ represents a cycle, it is immediate that $\sum a_j [F_i(c_p(1)) - F_i(c_p(0))] = 0$. Hence

$$\begin{aligned} \ell_i^{\rho'}(\alpha) - \ell_i^\rho(\alpha) &= \sum [\pm 1]_p a_p \varepsilon_i(c_p) \\ &= \tau(\alpha, \varepsilon_i) \end{aligned}$$

where the coefficient $[\pm 1]_p$ takes the values $+1$ or -1 depending on whether the transverse arc c_p is positively or negatively oriented for the transverse orientation of $\hat{\lambda}$. This achieves the proof of Theorem 40. \square

Acknowledgments. I would like to thank Anna Wienhard, Bill Goldman, Dick Canary and Olivier Guichard, for the constant support and interest they showed in this work, as well as the whole GEAR community. Last but not least, I would like to thank my advisor, Francis Bonahon, for teaching me, with unlimited patience and enthusiasm, the powerful techniques of transverse structures for geodesic laminations.

REFERENCES

- [Bon₁] Francis Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. **5** (1996), pp. 233–297.
- [Bon₂] Francis Bonahon, *Geodesic laminations with transverse Hölder distributions*, Ann. Sci. Ecole Norm. Sup. **30** (1997), pp. 205–240.
- [Bon₃] Francis Bonahon, *Transverse Hölder distributions for geodesic laminations*, Topology, Vol. **36** (1997), pp. 103–122.
- [Bon₄] Francis Bonahon, *Geodesic laminations on surfaces*, in: *Laminations and Foliations in Dynamics, Geometry and Topology* (M. Lyubich, J.W. Milnor, Y.N. Minsky eds.), Contemporary Mathematics vol. **269**, American Math. Soc. (2001), pp. 1–37.
- [BonDr₁] Francis Bonahon, Guillaume Dreyer, *Parametrizing Hitchin components*, submitted, available at [arXiv:1209.3526](https://arxiv.org/abs/1209.3526).
- [BonDr₂] Francis Bonahon, Guillaume Dreyer, *Hitchin representations and geodesic laminations*, in preparation.
- [ChGo] Suhyoung Choi, William M. Goldman, *Convex real projective structures on closed surfaces are closed*, Proc. Amer. Math. Soc. **118** (1993), pp. 657–661.
- [Dr₁] Guillaume Dreyer, *Length functions of Hitchin representations*, submitted, available at [arXiv:1106.6310](https://arxiv.org/abs/1106.6310).
- [Dr₂] Guillaume Dreyer, *The space of Anosov representations along a geodesic lamination*, in preparation.
- [FLP] Albert Fathi, François Laudenbach, Valentin Poénaru, *Thurston's Work on Surfaces*, translated by Djun Kim and Dan Margarita, Princeton University Press, Princeton (2012).
- [FoGo] Vladimir Fock, Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), pp. 1–211.
- [Ghy] Etienne Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'après M. Gromov*, Prog. Math., Vol. **83** (1990), Birkhäuser.
- [Gol] William M. Goldman, *Topological components of spaces of representations*, Invent. Math., Vol. **93** (1988), pp. 557–607.
- [Gro] Mikhail Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ. Vol. **8** (1987), Springer, New York, pp. 75–263, .
- [Gui] Olivier Guichard, *Composantes de Hitchin et représentations hyperconvexes de groupes de surface*, J. Differential Geom., Vol. **80** (2008), pp. 391–431.
- [GuiW₁] Olivier Guichard, Anna Wienhard, *Anosov representations: Domains of discontinuity and applications*, preprint (2011), available at [arXiv:1108.0733v2](https://arxiv.org/abs/1108.0733v2).
- [GuiW₂] Olivier Guichard, Anna Wienhard, *Topological invariants of Anosov representations*, Topology, Vol. **3** (2010), pp. 578–642.
- [Hit] Nigel J. Hitchin, *Lie Groups and Teichmüller space*, Topology, Vol. **31** (1992), pp. 449–473.
- [Ker] Steven P. Kerckhoff, *The Nielsen realization problem*, Annals of Mathematics (second series) **117** (2) (1983), pp. 235–265.
- [La] François Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math., Vol. **165** (2006), pp. 51–114.
- [Lu₁] Georges Lusztig, *Total positivity in reductive groups*, Lie Theory and Geometry, Progr. Math., vol. **123**, Birkhäuser Boston, Boston, MA (1994), pp. 531–568.
- [Lu₂] Georges Lusztig, *Total positivity in partial flag manifolds*, Representation Theory **2** (1998), pp. 70–78 (electronic).
- [Mar] Gregori A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. **17** (1991), Springer-Verlag, Berlin.
- [PeH] Robert C. Penner, John L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies vol. **125**, Princeton University Press, Princeton (1992).
- [deRh] Georges de Rham, *Differentiable manifolds: forms, currents, harmonic forms*, Grundlehren der mathematischen Wissenschaften, Vol. **266** (1984), Springer-Verlag, Berlin.
- [RuSu] David Ruelle, Dennis Sullivan, *Currents, flows and diffeomorphisms*, Topology, Vol. **14** (1975), pp. 319–327.
- [Th₁] William P. Thurston, *The geometry and topology of 3-manifolds*, Princeton lecture notes (1978–1981), available at <http://library.msri.org/books/gt3m/>.
- [Th₂] William P. Thurston, *Minimal stretch maps between hyperbolic surfaces*, unpublished preprint, available at [arXiv:math/9801039](https://arxiv.org/abs/math/9801039).

- [We] André Weil, *On discrete subgroups of Lie groups*, Ann. Math. **72** (1960), 369–384, **75** (1962), pp. 578–602.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, 255 HURLEY HALL, NOTRE DAME, IN 46556, U.S.A.

E-mail address: dreyfactor@gmail.com, gdreyer@alumni.usc.edu, gdreyer@nd.edu